

Hausdorff spectra of the local Hölder exponent of Weierstrass-type functions

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1 Introduction

Let $\Sigma_\ell = \{0, \dots, \ell - 1\}$ for $\ell \in \mathbb{N}$. Given disjoint subintervals $(I_i)_{i \in \Sigma_\ell}$ of $[0, 1]$, we call a map $\tau : \bigcup_{i \in \Sigma_\ell} I_i \rightarrow [0, 1]$ a cookie-cutter map if each restriction $\tau|_{I_i^\circ}$ can be extended to a uniformly expanding C^{1+} -diffeomorphism on some neighbourhood of $\overline{I_i}$, where A° and \overline{A} denote the interior and the closure of a subset $A \subseteq [0, 1]$, respectively. Here, C^{1+} stands for $C^{1+\varepsilon}$ for some $\varepsilon > 0$. In addition, 'uniformly expanding' means that

$$\inf_{i \in \Sigma_\ell} \inf_{x \in I_i} |(\tau|_{I_i})'(x)| > 1. \quad (1)$$

The intervals $(I_i)_{i \in \Sigma_\ell}$ are called the monotonicity intervals of τ . Furthermore, the set of singular points and the 'repeller' of τ are respectively defined as

$$\mathcal{N} = \bigcup_{n \in \mathbb{N}} \tau^{-n} \{0, 1\} \quad \text{and} \quad \mathcal{J} = \bigcap_{n \in \mathbb{N}} \overline{\tau^{-n} [0, 1]}. \quad (2)$$

Extending τ to some endpoints of the monotonicity intervals (and modifying the corresponding intervals, too) if necessary, we may assume that τ is defined on the whole \mathcal{J} . Observe that $\tau(\mathcal{J}) = \mathcal{J} = \tau^{-1}(\mathcal{J})$. In the following, we regard τ as defined on \mathcal{J} , i.e. $\tau : \mathcal{J} \rightarrow \mathcal{J}$. Note that the classical choice is $\tau(x) = \ell x \bmod 1$ for $\ell \in \mathbb{N} \setminus \{1\}$. In this case, $\mathcal{J} = [0, 1]$ and the monotonicity intervals $(I_i)_{i \in \Sigma_\ell}$ form a partition of $[0, 1]$.

Now, the Weierstrass-type function is defined as

$$W(x) = \sum_{n=0}^{\infty} \lambda(x) \lambda(\tau x) \cdots \lambda(\tau^{n-1} x) g(\tau^n x)$$

for $x \in \mathcal{J}$, where we assume that

- $\lambda : [0, 1] \rightarrow (0, 1)$ is continuous and piecewise C^1 ,
- $g : [0, 1] \rightarrow \mathbb{R}$ is continuous and piecewise C^1 , and
- we have the partial hyperbolic condition

$$\inf_{i \in \Sigma_\ell} \inf_{x \in I_i} |(\tau|_{I_i})'(x)| \lambda(x) > 1. \quad (3)$$

In this note, 'piecewise' is always related to the monotonicity intervals $(I_i)_{i \in \Sigma_\ell}$ of τ . In addition, we also consider the randomised version of the Weierstrass-type function

$$W_{\boldsymbol{\vartheta}}(x) = \sum_{n=0}^{\infty} \lambda(x) \lambda(\tau x) \cdots \lambda(\tau^{n-1} x) g(\tau^n x + \vartheta_n), \quad (4)$$

where $\boldsymbol{\vartheta} = (\vartheta_n)_{n \in \mathbb{N}_0} \in [0, 1]^{\mathbb{N}_0}$ is used as the random element. Note that W can be regarded as $W_{\mathbf{0}}$ for $\mathbf{0} = (0, 0, \dots)$.

It is clear that $W_{\boldsymbol{\vartheta}} : \mathcal{J} \rightarrow \mathbb{R}$ is continuous. Indeed, the function has a canonical continuous extension on $[0, 1]$.

Lemma 1. *$W_{\boldsymbol{\vartheta}}$ has a continuous extension $W_{\boldsymbol{\vartheta}}^{\text{ext}} : [0, 1] \rightarrow \mathbb{R}$ for each $\boldsymbol{\vartheta}$, which is C^1 on $[0, 1] \setminus (\mathcal{J} \cup \mathcal{N})$.*

Proof. This is an immediate consequence of the identification between $[0, 1]$ and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Namely, let the map $\tilde{\tau} : [0, 1] \rightarrow [0, 1]$ be defined by

$$\tilde{\tau}(x) := \begin{cases} \tau(x) & \text{if } x \in \bigcup_{i \in \Sigma_\ell} I_i \\ 0 & \text{otherwise} \end{cases}.$$

Then, we define $W_{\boldsymbol{\vartheta}}^{\text{ext}}$ by the formula (4), replacing τ by $\tilde{\tau}$. Since all the summands are continuous on $[0, 1]$, the series converges absolutely to a continuous function.

To see the claimed C^1 -regularity, let $x \in [0, 1] \setminus (\mathcal{J} \cup \mathcal{N})$. Then, for some $n \in \mathbb{N}$, we have $x \in U := [0, 1] \setminus \overline{\tau^{-n}[0, 1]}$. Observe that U is a neighbourhood of x such that

$$W_{\boldsymbol{\vartheta}}(u) = \sum_{j=0}^{n-1} \lambda(u) \cdots \lambda(\tilde{\tau}^{j-1} u) g(\tilde{\tau}^j u + \theta_j) + \frac{\lambda(u) \cdots \lambda(\tilde{\tau}^n u)}{1 - \lambda(0)} \sum_{k=0}^{\infty} (\lambda(0))^k g(\theta_k)$$

for $u \in U$. Moreover, as $x \notin \mathcal{N}$, there is a sub-neighbourhood $x \in V \subseteq U$ such that $\tilde{\tau}, \dots, \tilde{\tau}^n$ are C^1 on V . Thus $W_{\boldsymbol{\vartheta}}$ is C^1 in x . \square

While the extended part of $W_{\boldsymbol{\vartheta}}^{\text{ext}}$ is smooth enough from the dimension theoretic point of view, the $W_{\boldsymbol{\vartheta}}$ itself is typically nowhere differentiable on \mathcal{J} as shown in [Bed89a] so that fractal dimensions of its graph

$$\mathcal{GW}_{\boldsymbol{\vartheta}} = \{(x, W_{\boldsymbol{\vartheta}}(x)) : x \in \mathcal{J}\}$$

are possibly greater than those of its domain \mathcal{J} . In the case $\mathcal{J} = [0, 1]$ and $\tau' > 0$, the box dimension of the graph of W was proved in [Bed89a], while its Hausdorff dimension in the classical case is proved in [She15], [Bá15] and [Kel14]. Related to this problem, the randomised function $W_{\boldsymbol{\vartheta}}$ was studied in [Hun98] and [MW12].

In this note, we study the box dimension of the randomised $W_{\boldsymbol{\vartheta}}$, the Hausdorff spectrum of the local Hölder exponent of both W and a lifted version for the randomised $W_{\boldsymbol{\vartheta}}$. Our results also related to [Bed89b], in which the local Hölder exponent of W is studied, especially with respect to the Lebesgue measure. In addition, related to this problem, we also study the Hausdorff dimension of lifted Gibbs measures on $\mathcal{GW}_{\boldsymbol{\vartheta}}$.

2 Preliminaries

Our basis dynamics will be referred to as (\mathcal{J}, τ) rather than $([0, 1], \tau)$ since the map is only defined on $\bigcup_{i \in \Sigma_\ell} I_i \subseteq [0, 1]$. Clearly, \mathcal{J} is a compact metric space with respect to the induced metric. Recall that the monotonicity intervals $(I_i)_{i \in \Sigma_\ell}$ are, however, subsets of $[0, 1]$.

Here are further conventions.

- Let $\kappa(x) := i$ for $x \in I_i$ and $i \in \{0, \dots, \ell - 1\}$. In addition, let

$$[x]_n := (\kappa(x), \kappa(\tau x), \dots, \kappa(\tau^{n-1}x))$$

for $x \in \mathcal{J}$ and $n \in \mathbb{N}$.

- The inverse branches of τ are defined as C^{1+} -diffeomorphisms $\rho_i : [0, 1] \rightarrow \bar{I}_i$ for $i \in \Sigma_\ell$ by extending $(\tau|_{I_i})^{-1}$, continuously.
- The n -th monotonicity interval of $x \in \mathcal{J}$ is the subinterval $I_n(x) \subseteq [0, 1]$ defined by

$$I_n(x) := I_{\kappa(x)} \cap \tau^{-1} I_{\kappa(\tau x)} \cap \dots \cap \tau^{-(n-1)} I_{\kappa(\tau^{n-1}x)},$$

where all preimages are related to the original map $\tau : \bigcup_{i \in \Sigma_\ell} I_i \rightarrow [0, 1]$.

- Given any function $\phi : \mathcal{J} \rightarrow \mathbb{R}$ we simply write

$$\phi_n(x) := \sum_{k=0}^{n-1} \phi(\tau^k x) \quad \text{and} \quad \phi^n(x) := \prod_{k=0}^{n-1} \phi(\tau^k x)$$

for $x \in \mathcal{J}$ and $n \in \mathbb{N}_0$.

- Given Borel measurable subset $A \subseteq \mathbb{R}^d$, let $\mathcal{P}(A)$ denote the set of all Borel probability measures on A . Moreover, let $\mathcal{P}_\tau(\mathcal{J})$ denote the set of all τ -invariant Borel measures ν on \mathcal{J} , where the τ -invariance means $\nu \circ \tau^{-1} = \nu$.
- Given piecewise C^1 -function $\phi : \bigcup_{i \in \Sigma_\ell} I_i \rightarrow \mathbb{R}$, we simply write its derivative as ϕ' , considering only differentiable points or, if necessary, extending the derivatives continuously to the endpoints of monotonicity intervals. Note that ϕ' is defined at least on $\mathcal{J} \setminus \mathcal{N}$, uniquely.

Gibbs measure and topological pressure

In this note, Gibbs measures are always assumed to have Hölder continuous potentials, that is, $\nu \in \mathcal{P}_\tau(\mathcal{J})$ is a Gibbs measure if for some constants $C_\phi > 0$ and P_ϕ and a Hölder continuous function $\phi : \mathcal{J} \rightarrow \mathbb{R}$ we have that

$$C_\phi^{-1} \leq \frac{\nu(I_n(x))}{e^{\phi_n(x) - P_\phi}} \leq C_\phi$$

holds for all $x \in \mathcal{J}$ and $n \in \mathbb{N}$. In addition, let $P(\phi)$ denote the topological pressure on (\mathcal{J}, τ) with respect to the potential ϕ . Recall the variational principle

$$P(\phi) = \sup_{\nu \in \mathcal{P}_\tau(\mathcal{J})} h_\tau(\nu) + \int \phi d\nu,$$

where $h_\tau(\nu)$ denotes the K-S-entropy. Furthermore, there exists a unique maximising measure of the above variational expression, which shall be referred to as the equilibrium state for the potential ϕ . See e.g. [Bar08] or [Pes97] for the details.

For the proof of the next lemma, consult [Kel98].

Lemma 2. *Let $\phi : \mathcal{J} \rightarrow \mathbb{R}$ be Hölder continuous. The equilibrium state for the potential ϕ is a Gibbs measure for the same potential, and vice versa, where we have $P_\phi = P(\phi)$. In addition, any Gibbs measure is ergodic and atom-free.*

Lifted measure

For each $\nu \in \mathcal{P}_\tau(\mathcal{J})$, its lift $\mu_\nu \in \mathcal{P}(\mathcal{J} \times \mathbb{R})$ is defined as

$$\mu_\nu := \nu \{x \in \mathcal{J} : (x, W_\nu(x)) \in \cdot\}.$$

Hausdorff dimension of measures

We follow the standard definitions of several dimensions in [Bar08]. Let μ be a Borel measure on a metric space (E, d) . The lower pointwise dimension of μ is defined as

$$\underline{d}_\mu(u) := \liminf_{r \rightarrow 0} \frac{\log \nu(B_r(u))}{\log r}$$

for each $u \in E$, where $B_r(u) := \{u' \in E : d(u, u') \leq r\}$ denote the closed balls. In addition, the Housdorff dimension of the measure μ , denoted by $\dim_H \mu$, is defined as

$$\dim_H \mu = \inf \{ \dim_H Z : Z \subseteq E \text{ s.t. } \mu^*(E \setminus Z) = 0 \},$$

where μ^* is the outer measure extension of μ .

The next lemma provides an alternative definition of $\dim_H \mu$.

Lemma 3 ([Bar08, Theorem 2.1.5 (3)]). *Let μ be a Borel measure on an Euclidean space E . Then $\dim_H \mu$ is the essential minimum of \underline{d}_μ with respect to μ .*

Local Hölder exponent

Assume that $E \subseteq \mathbb{R}$ be a subset equipped with the induced metric. The local Hölder exponent of a continuous function $\phi : E \rightarrow \mathbb{R}$ at $x \in E$ is defined by

$$\text{hol}_\phi(x) := \sup \left\{ \alpha \in [0, 1] : \inf_{r > 0} \sup_{u \in B_r(x)} \frac{|\phi(x) - \phi(u)|}{|x - u|^\alpha} < \infty \right\},$$

where $B_r(x) := \{u \in E : |x - u| \leq r\}$ denote the balls in E . In this note, we study the two types of level set

$$E_{\boldsymbol{\vartheta}, \alpha} := \{x \in [0, 1] : \text{hol}_{W_{\boldsymbol{\vartheta}}}(x) = \alpha\} \quad \text{and} \quad \tilde{E}_{\boldsymbol{\vartheta}, \alpha} := \{(x, W_{\boldsymbol{\vartheta}}(x)) \in [0, 1] \times \mathbb{R} : \text{hol}_{W_{\boldsymbol{\vartheta}}}(x) = \alpha\},$$

in terms of the Hausdorff spectrum, i.e. we study

$$\alpha \mapsto \dim_H E_{\boldsymbol{\vartheta}, \alpha} \quad \text{and} \quad \alpha \mapsto \dim_H \tilde{E}_{\boldsymbol{\vartheta}, \alpha}.$$

For simplicity, let $E_\alpha := E_{\mathbf{0}, \alpha}$ and $\tilde{E}_\alpha := \tilde{E}_{\mathbf{0}, \alpha}$ for $\mathbf{0} = (0, 0, \dots)$.

Basics from the thermodynamic formalisms

For each $q \in \mathbb{R}$, let $A_q \in \mathbb{R}$ be the number that is uniquely determined by

$$P(-A_q \log |\tau'| + q \log \lambda) = 0, \tag{5}$$

where $P(\cdot)$ denotes the topological pressure on (\mathcal{J}, τ) . As in [Bar08], we can define the following.

- Let $\alpha_q := -A'_q$.
- Let \mathcal{D} be the Legendre transform of the map $q \mapsto A_q$, i.e. $\mathcal{D}(\alpha) := \sup_{q \in \mathbb{R}} q\alpha + A_q$.
- Let $\mathbf{A} := (\alpha_{\min}, \alpha_{\max})$, where

$$\alpha_{\min} := \min_{\nu \in \mathcal{P}_\tau(\mathcal{J})} \frac{-\log \lambda d\nu}{\log |\tau'| d\nu} \quad \text{and} \quad \alpha_{\max} = \max_{\nu \in \mathcal{P}_\tau(\mathcal{J})} \frac{-\log \lambda d\nu}{\log |\tau'| d\nu}.$$

Recall that, if $\mathbf{A} \neq \emptyset$, the restriction of \mathcal{D} to the interval is a \mathbf{A} non-negative, smooth, and strictly concave function, which possesses a unique maximum point $\alpha_c \in \mathbf{A}$. In addition, we have

$$\mathcal{D}(\alpha_c) = 1 \quad \text{and} \quad \mathcal{D}'(\alpha_c) = 0.$$

In case $\mathbf{A} = \emptyset$, we define $\alpha_c := \alpha_{\min} = \alpha_{\max}$.

3 Main results

3.1 Degenerate case

First of all, we need to exclude a possible degenerate case, which happens e.g. by the trivial choice of $g \equiv 0$. We say that a function $\phi : \mathcal{J} \rightarrow \mathbb{R}$ is C^{1+} on \mathcal{J} if the derivative

$$x \mapsto \lim_{u \rightarrow x} \frac{\phi(x) - \phi(u)}{|x - u|}$$

exists and Hölder continuous, where the limit is considered over \mathcal{J} .

Lemma 4. *The Weierstrass-type function $W : \mathcal{J} \rightarrow \mathbb{R}$ is either*

- C^{1+} or
- nowhere differentiable.

Proof. In case $\mathcal{J} = [0, 1]$ and $\tau' > 0$, this is the content of [Bed89a, Section 5]. Observe that the same arguments can also apply generally. \square

In [Bed89a], the first case was referred to as the degenerate case. Slightly abusing this concept, in this note, we use the following definition. Given $\boldsymbol{\vartheta}$, the *degenerate case* is if the function $W_{\boldsymbol{\vartheta}}$ is Lipschitz continuous on \mathcal{J} . Simply, we say that $W_{\boldsymbol{\vartheta}}$ is *degenerate* in this case, or otherwise, that $W_{\boldsymbol{\vartheta}}$ is *non-degenerate*.

3.2 Results on the lifted measure

Following the notation of [MW12], we say that a function $g : [0, 1] \rightarrow \mathbb{R}$ satisfies the *critical point hypothesis* if $g \in C^\infty([0, 1])$ and there is some number $r_0 \in \mathbb{N}$ such that the orders of critical points of the functions $g(a + \cdot) - cg$ are strictly less than r_0 for any $a \in (0, 1)$ and $c \in \mathbb{R}$. For example, any non-vanishing trigonometric polynomial satisfies this condition.

Theorem 1. *Suppose that $\nu \in \mathcal{P}_\tau(\mathcal{J})$ is a Gibbs measure and that $\boldsymbol{\vartheta} \in [0, 1]^{\mathbb{N}_0}$ is i.i.d. to the uniform distribution on $[0, 1]$. If g satisfies the critical point hypothesis, then almost surely we have*

$$\dim_H \mu_{\boldsymbol{\vartheta}} = \min \left\{ \dim_H \nu + 1 + \frac{\int \log \lambda \, d\nu}{\int \log |\tau'| \, d\nu}, \frac{h_\tau(\nu)}{\int \log |\tau'| \, d\nu} \right\}.$$

Remark 5. In the proof of [MW12, Proposition 2.3], in order to establish a lower bound of $\dim_H \mathcal{G}W_{\boldsymbol{\vartheta}}$ in case $\mathcal{J} = [0, 1]$ and $\tau' > 0$, the authors actually proved the following: The Hausdorff dimension of the lift of a Gibbs measure ν_ϕ for a potential ϕ is almost surely bounded from below by the number s chosen by

$$P((s - 1) \log |\tau'| + (\phi - P(\phi)) - \log \lambda) = 0.$$

This yields the sharp lower bound of the dimension of the lifted measure for $\phi = (1 - s) \log |\tau'| + \log \lambda$, however, does quite rarely for any other ϕ , see [Jin11, Remark 5]. On the other hand, Theorem 1 provides the exact value of the dimension.

3.3 Results on the dimension of the graph

Let $s_1, s_2 \in \mathbb{R}$ be the unique zeros of the Bowen equations

$$P((1 - s_1) \log |\tau'| + \log \lambda) = 0 \quad \text{and} \quad P(s_2 \log \lambda) = 0, \tag{6}$$

where $P(\cdot)$ denotes again the topological pressure on (\mathcal{J}, τ) . The first result includes a generalisation of the result of [Bed89a].

Theorem 2. *Suppose that W is non-degenerate. Then we have*

$$\dim_B \mathcal{G}W = s_1 \quad \text{and} \quad \dim_H \mathcal{G}W \leq \min\{s_1, s_2\}.$$

Corollary 6. *Suppose that W is non-degenerate. If $\dim_B \mathcal{G}W < 1$, or equivalently, if $P(\log \lambda) < 0$, then $\dim_H \mathcal{G}W < \dim_B \mathcal{G}W$.*

The equality $\dim_H \mathcal{G}W = \min\{s_1, s_2\}$ in general is an open problem, for which we do not know any counter example. The next result gives an answer for a randomised version.

Theorem 3. *Suppose that $\vartheta \in [0, 1]^{\mathbb{N}_0}$ is i.i.d. to the uniform distribution on $[0, 1]$. If g satisfies the critical point hypothesis, then almost surely*

$$\dim_B \mathcal{G}W_{\vartheta} = s_1 \quad \text{and} \quad \dim_H \mathcal{G}W_{\vartheta} = \min\{s_1, s_2\}$$

3.4 Results on the local Hölder spectra

Firstly, we present a very general deterministic result on the spectrum $\alpha \mapsto \dim_H E_{\alpha}$.

Theorem 4. *Suppose that W is non-degenerate. In case $A = \emptyset$, we have*

$$E_{\alpha} = \begin{cases} \mathcal{J} & \text{if } \alpha = \alpha_c \\ \emptyset & \text{otherwise} \end{cases}.$$

In case $A \neq \emptyset$, we have

$$\dim_H E_{\alpha} = \mathcal{D}(\alpha)$$

for all $\alpha \in A$. In this case, for each $\alpha \in A$, there is a Gibbs measure ν_{α} such that $\nu_{\alpha}(E_{\alpha}) = 1$ and

$$\dim_H \nu_{\alpha} = \mathcal{D}(\alpha) = \frac{h_{\tau}(\nu_{\alpha})}{\int \log |\tau'| d\nu_{\alpha}} \quad \text{and} \quad \alpha = \frac{-\int \log \lambda d\nu_{\alpha}}{\int \log |\tau'| d\nu_{\alpha}}.$$

Remark 7. The first case in the above theorem happens e.g. if we choose $\lambda = |\tau'|^{-\theta}$ for some constant $\theta \in (0, 1)$. Clearly, then $\alpha_c = \theta$. In particular, Theorem 4 implies [Tod15, Theorem 1], i.e. that the Lebesgue measure of E_{θ} is one.

Remark 8. In general, $E_{\alpha} \neq \emptyset$ if and only if $\alpha \in \bar{A}$. However, the values of $\dim_H E_{\alpha}$ for $\alpha \in \partial A$ are not known.

Now, we consider the spectrum of the lifted level set $\alpha \mapsto \dim_H \tilde{E}_{\alpha}$. The following canonical upper bound follows from Theorem 4 in view of [Jin11, Theorem 1], immediately.

Lemma 9. *We have*

$$\dim_H \tilde{E}_{\vartheta, \alpha} \leq \min \left\{ \dim_H E_{\vartheta, \alpha} + 1 - \alpha, \frac{\dim_H E_{\vartheta, \alpha}}{\alpha} \right\}$$

for all $\alpha \in \mathbb{R}$ and $\vartheta \in [0, 1]^{\mathbb{N}_0}$.

The next result on the randomised case is an application of Theorem 1, which suggests the canonical representation for the lifted spectrum.

Theorem 5. *Suppose that ϑ is i.i.d. to the uniform distribution on $[0, 1]$. Furthermore, suppose that g satisfies the critical point hypothesis. Then almost surely we have:*

- *If $A = \emptyset$, then we have*

$$\tilde{E}_{\vartheta, \alpha} = \begin{cases} \mathcal{G}W_{\vartheta} & \text{if } \alpha = \alpha_c \\ \emptyset & \text{otherwise} \end{cases}.$$

- If $A \neq \emptyset$, then we have

$$\dim_H \tilde{E}_{\mathcal{A}, \alpha} = \min \left\{ \mathcal{D}(\alpha) + 1 - \alpha, \frac{\mathcal{D}(\alpha)}{\alpha} \right\}$$

for all $\alpha \in A$. In particular,

$$\dim_H \mathcal{GW}_{\mathcal{A}} = \max_{\alpha \in A} \min \left\{ \mathcal{D}(\alpha) + 1 - \alpha, \frac{\mathcal{D}(\alpha)}{\alpha} \right\}.$$

Remark 10. If W is non-degenerate and $A \neq \emptyset$, one can show

$$\dim_B \mathcal{GW} = \sup_{\alpha \in A} \{ \dim_H E_{\alpha} + 1 - \alpha \}.$$

Remark 11. Note that a deterministic example with Takagi function for which $\dim_H \tilde{E}_{\alpha} = \tilde{\mathcal{D}}(\alpha)$ holds for all α in a large neighbourhood of $\tilde{\alpha}_c$ can be constructed.

4 Fundamental techniques

4.1 Some properties of (\mathcal{J}, τ)

Lemma 12. *There is a constant $D > 0$ such that*

$$\left| \frac{(\tau^n)'(x)}{(\tau^n)'(v)} \right| \in [D^{-1}, D] \quad |(\tau^n)'(x) \cdot |I_n(u)| \in [D^{-1}, D] \quad \text{and} \quad \frac{|\rho'_{[x]_n}(v)|}{|I_n(x)|} \in [D^{-1}, D]$$

holds for all $u \in I_n(x)$, $x, v \in [0, 1]$ and $n \in \mathbb{N}$.

Furthermore, there is a number $\delta_0 > 0$ such that

$$\delta_0 \leq \frac{|I_{n+1}(x)|}{|I_n(x)|}$$

for all $x \in \mathcal{J}$ and $n \in \mathbb{N}$.

Proof. Recall that τ' is piecewise ε -Hölder continuous so that $\log \tau'$ is also. The first one is classical, i.e. it follows as

$$\frac{(\tau^n)'(x)}{(\tau^n)'(v)} = e^{\sum_{j=0}^{n-1} (\log \tau'(\tau^j x) - \log \tau'(\tau^j v))} \leq e^{C_0 \sum_{j=0}^{n-1} |I_{n-j}(\tau^j x)|^{\varepsilon}} \leq e^{C_0/(1-\|1/\tau'\|_{\infty}^{\varepsilon})},$$

where C_0 is the ε -Hölder constant of $\log \tau'$. Note that we used also the fact $|I_n(x)| \leq \|1/\tau'\|_{\infty}^n$ for $x \in [0, 1]$ and $n \in \mathbb{N}$, which follows by the mean value theorem. The second one is due to the mean value theorem and the last one is only a rewriting of the second one. \square

Lemma 13. *Let $\nu \in \mathcal{P}_{\tau}([0, 1])$ be a Gibbs measure. Suppose that $x \in [0, 1]$ is eventually periodic, i.e. there are $M, p \in \mathbb{N}$ such that $\tau^{M+p}x = \tau^Mx$. Then, $\lim_{N \rightarrow \infty} \nu(I_N(x)) = 0$ converges exponentially fast.*

Proof. Let C_{ϕ} are ϕ be in the definition of Gibbs measure. Let $x \in [0, 1]$ and $M, p \in \mathbb{N}$ such that $\tau^{M+p}x = \tau^Mx$. In view of $\phi_{pk}(\tau^Mx) = k\phi_p(\tau^Mx)$, the Gibbs property asserts that

$$C_{\phi}^{-1} \leq \frac{\nu(I_{M+pk}(x))}{e^{\phi_{M-1}(x) - (M-1)P_{\phi}} e^{k(\phi_p(\tau^Mx) - pP_{\phi})}} \leq C_{\phi}$$

holds for all $k \in \mathbb{N}$. This, in turn, implies that $\phi_p(\tau^Mx) - pP_{\phi} < 0$ since ν is non-atomic. Thus, $\lim_{k \rightarrow \infty} \nu(I_{M+pk}(x)) = 0$ converges exponentially fast. This finishes the proof in view of the monotonicity of $N \mapsto \nu(I_N(x))$. \square

Given an interval I , let $|I|$ and ∂I denote its length and the set of its endpoints, respectively. That is, $|I| := b - a$ and $\partial I := \{a, b\}$, where $\bar{I} =: [a, b]$. Recall that the distance between a set and a point is defined as $\text{dist}(A, x) := \inf\{|x - v| : v \in A\}$ for $A \subseteq [0, 1]$ and $x \in [0, 1]$.

Lemma 14. *Let $\nu \in \mathcal{P}_\tau([0, 1])$ be a Gibbs measure. Then we have*

$$\lim_{n \rightarrow \infty} \frac{\log \text{dist}(\partial I_n(x), x)}{\log |I_n(x)|} = 1$$

for ν -a.a. $x \in [0, 1]$.

Proof. In view of $\text{dist}(\partial I_n(x), x) \leq |I_n(x)|$, we only need to show that the limit superior of the left hand side is bounded by 1 from above.

Let $\alpha \in (0, 1)$ be arbitrary. Then, let $M_n := \lfloor \alpha n \rfloor$. Furthermore, let

$$E_n := \tau^{n-M_n} (I_{M_n}(0) \cup I_{M_n}(1))$$

for $n \in \mathbb{N}$. Observe that for $x \in [0, 1] \setminus E_n$ we have

$$\text{dist}(\partial I_n(x), x) \geq \min \{ |\rho_{[x]_{n-M_n}}(I_{M_n}(0))|, |\rho_{[x]_{n-M_n}}(I_{M_n}(1))| \}.$$

Since by Lemma 12

$$\begin{aligned} |\rho_{[x]_{n-M_n}}(I_{M_n}(i))| &\geq (\min \rho'_{[x]_{n-M_n}}) |I_{M_n}(i)| \\ &\geq D^{-1} |I_{n-M_n}(x)| |I_{M_n}(i)| \\ &\geq D^{-2} |I_{n-M_n}(x)| (\min(1/\tau'))^{M_n} \end{aligned}$$

for $i \in \{0, 1\}$, we have

$$\text{dist}(\partial I_n(x), x) \geq D^{-2} |I_{n-M_n}(x)| (\min(1/\tau'))^{M_n}.$$

Thus, in view of Birkhoff's ergodic theorem and Lemma 12, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\log \text{dist}(\partial I_n(x), x)}{\log |I_n(x)|} &\leq \lim_{n \rightarrow \infty} \frac{\log |I_{n-M_n}(x)|}{\log |I_n(x)|} + \lim_{n \rightarrow \infty} \frac{M_n}{\log |I_n(x)|} \log(\min(1/\tau')) \\ &= (1 - \alpha) + \alpha \frac{\log(\min(1/\tau'))}{-\int \log \tau' d\nu} \end{aligned} \quad (7)$$

for ν -a.a. $x \in [0, 1] \setminus \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} E_n$. On the other hand, by Borel–Cantelli lemma we have $\nu \left(\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} E_n \right) = 0$ since

$$\nu(E_n) = \nu(I_{M_n}(0) \cup I_{M_n}(1)) \leq \nu(I_{M_n}(0)) + \nu(I_{M_n}(1))$$

is summable by Lemma 13. Therefore, (7) holds ν -a.a. $x \in [0, 1]$. Finally, letting $\alpha \searrow 0$ finishes the proof. \square

4.2 Properties of Gibbs measures

Lemma 15. *If ν is a Gibbs measure for ϕ , then*

$$\dim_H \nu = \frac{h_\tau(\nu)}{-\int \log \tau' d\nu}.$$

Proof. Let $B_r(x) := \{x' \in [0, 1] : |x - x'| \leq r\}$ be balls for $x \in [0, 1]$ and $r > 0$. Observe that, by Lemmas 12 and 14,

$$B_{\text{dist}(\partial I_n(x), x)}(x) \subseteq I_n(x) \subseteq B_{|I_n(x)|}(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log |I_n(x)|}{\log |I_{n+1}(x)|} = 1$$

hold for all $x \in [0, 1]$ and $n \in \mathbb{N}$, we have

$$\underline{d}_\nu(x) = \liminf_{n \rightarrow \infty} \frac{\log \nu(I_n(x))}{\log |I_n(x)|}$$

for ν -a.a. $x \in [0, 1]$. Thus, the claim follows from this in view of Shannon–McMillan–Breiman theorem and Birkhoff’s ergodic theorem together with Lemma 12. \square

4.3 Properties of local Hölder exponent

The next lemma give a useful formula for the local Hölder exponent that is true in general situations.¹

Lemma 16. *Assume that $E \subseteq \mathbb{R}$ be a subset equipped with the induced metric. For any continuous function $\phi : E \rightarrow \mathbb{R}$, we have*

$$\text{hol}_\phi(x) = \lim_{r \searrow 0} \inf_{u \in B_r(x)} \frac{\log |\phi(x) - \phi(u)|}{\log r}$$

for all $x \in E$.

Proof. Given $x \in E$, let \tilde{h} denote the value above on the right hand side. Let $r_k \searrow 0$ be a sequence such that

$$\inf_{u \in B_{r_k}(x)} \frac{\log |\phi(x) - \phi(u)|}{\log r_k} \in B_{1/k}(\tilde{h})$$

holds for all $r \in (0, r_k)$. Observe that

$$\begin{aligned} \inf_{r > 0} \sup_{u \in B_r(x)} \frac{|\phi(x) - \phi(u)|}{|x - u|^{\tilde{h}-1/k}} &= \inf_{r \in (0, r_k)} \sup_{u \in B_r(x)} \frac{|\phi(x) - \phi(u)|}{|x - u|^{\tilde{h}-1/k}} \\ &\leq \inf_{r \in (0, r_k)} \sup_{u \in B_r(x)} \underbrace{\sup_{v \in B_{|x-u|}(x)} \frac{|\phi(x) - \phi(v)|}{|x - u|^{\tilde{h}-1/k}}}_{\leq 1} \leq 1 \end{aligned}$$

and

$$\sup_{u \in B_{r_j}(x)} \frac{|\phi(x) - \phi(u)|}{|x - u|^{\tilde{h}+2/k}} \geq \sup_{u \in B_{r_j}(x)} \frac{|\phi(x) - \phi(u)|}{|x - u|^{\tilde{h}+2/j}} \geq \sup_{u \in B_{r_j}(x)} \frac{r_j^{\tilde{h}+1/j}}{r_j^{\tilde{h}+2/j}} = \inf_{r \in (0, r_j)} r^{-1/j} = r_j^{-1}$$

for all $k, j \in \mathbb{N}$ with $j \geq k$. Since these together imply $\tilde{h} - 1/k \leq \text{hol}_\phi(x) \leq \tilde{h} + 2/k$ for all $k \in \mathbb{N}$, the proof is finished. \square

The next is yet another formula of $\text{hol}_\phi(x)$ which is defined via the dynamics (\mathcal{J}, τ) .

Lemma 17. *For any continuous function $\phi : \mathcal{J} \rightarrow \mathbb{R}$, we have*

$$\text{hol}_\phi(x) = \liminf_{n \rightarrow \infty} \inf_{u \in \mathcal{J} \cap I_n(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|}$$

for all $x \in \mathcal{J} \setminus \mathcal{N}$.

¹Note that [JS15, Lemma 5.1] asserts the same thing but its proof seems not to be precise. Indeed, their proof looks valid for another formula $\text{hol}(x) = \lim_{r \searrow 0} \inf_{u \in B_r(x)} \frac{\log |W(x) - W(u)|}{\log |x - u|}$.

Proof. Let $D > 0$ be the constant from Lemma 12. In view of the lemma, let $\delta := D^{-2} \min\{|I_2(0)|, |I_2(1)|\}$ so that for any $x \in \mathcal{J}$ and $n \in \mathbb{N}$ we have the implication:

$$\text{dist}(\partial(I_n(x)), \{x\}) < \delta |I_n(x)| \implies \tau^n x \in I_2(0) \cup I_2(1).$$

In addition, let

$$\mathcal{J}_0 := \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \{x \in \mathcal{J} : \text{dist}(\partial(I_n(x)), \{x\}) \geq \delta |I_n(x)|\}.$$

For $x \in \mathcal{J}_0$, since the inequalities

$$\inf_{u \in B_{|I_n(x)|}} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|} \leq \inf_{u \in \mathcal{J} \cap I_n(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|} \leq \inf_{u \in B_{\delta |I_n(x)|}(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|}$$

hold for infinitely many $n \in \mathbb{N}$, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \inf_{u \in \mathcal{J} \cap I_n(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|} &= \liminf_{n \rightarrow \infty} \inf_{u \in B_{|I_n(x)|}(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|} \\ &= \liminf_{r \searrow 0} \inf_{u \in B_r(x)} \frac{\log |\phi(x) - \phi(u)|}{\log r}, \end{aligned}$$

where the last equality is due to the fact $|I_n(x)| \searrow 0$ with $\lim_{n \rightarrow \infty} \frac{\log |I_n(x)|}{\log |I_{n+1}(x)|} = 1$. Together with Lemma 16, it follows that

$$\text{hol}_\phi(x) = \liminf_{n \rightarrow \infty} \inf_{u \in \mathcal{J} \cap I_n(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|}$$

for all $x \in \mathcal{J}_0$.

Finally, we show $\mathcal{J} \setminus \mathcal{N} \subseteq \mathcal{J}_0$, or equivalently, $\mathcal{J} \setminus \mathcal{J}_0 \subseteq \mathcal{N}$. Let $x \in \mathcal{J} \setminus \mathcal{J}_0$, i.e. there is some $N \in \mathbb{N}$ such that

$$\text{dist}(\partial(I_n(x)), \{x\}) < \delta |I_n(x)|$$

holds for all $n \geq N$. In view of the choice of $\delta > 0$, this implies that $\tau^n x \in I_2(0) \cup I_2(1)$ for all $n \geq N$, which is only true when $\tau^N x \in \{0, 1\}$. Thus, $x \in \mathcal{N}$. \square

5 Proof of Theorem 1

5.1 Upper bound

Lemma 18. *Let $\nu \in \mathcal{P}_\tau([0, 1])$ be a Gibbs measure. Then we have*

$$\dim_H \mu_\vartheta \leq \min \left\{ \dim_H \nu + 1 + \frac{\int \log \lambda d\nu}{\int \log \tau' d\nu}, \frac{h_\tau(\nu)}{\int \log \tau' d\nu} \right\}$$

for all $\vartheta \in [0, 1]^{\mathbb{N}_0}$.

Proof. Let $\vartheta \in [0, 1]^{\mathbb{N}_0}$ be given. As we will check in Lemma 29 later, the inequality of [MW12, Proposition 3.1] is valid in our slight different setting, i.e. there is a constant $\bar{C} > 0$ such that

$$\sup_{v \in \mathcal{J} \cap I_n(x)} |W_\vartheta(x) - W_\vartheta(v)| \leq \bar{C} \lambda^n(x) \quad (8)$$

holds for all $x \in [0, 1]$ and $n \in \mathbb{N}$. Let $\varepsilon \in (0, e^{-\int \log \tau' d\nu})$ be chosen arbitrarily. Let $D_N \subseteq [0, 1]$ denote the set of those $x \in [0, 1]$ which satisfy: For all $n \geq N$ hold all of

- $\left| \frac{\log \lambda^n(x)}{n} - \int \log \lambda d\nu \right| < \varepsilon,$

- $\left| \frac{-\log \nu(I_n(x))}{n} - h_\tau(\nu) \right| < \varepsilon$ and
- $\left| \frac{-\log |I_n(x)|}{n} - \int \log \tau' d\nu \right| < \varepsilon$.

By Birkhoff's ergodic theorem, Shannon-McMillan-Breiman theorem and Lemma 12, we have $\nu(\bigcup_N D_N) = 1$. Now, we consider the lifted set $G_N := \{(x, W_\theta(x)) : x \in D_N\}$. For each $n \geq N$ let $\mathcal{U}_n^N := \{I_n(x) : x \in D_N\}$. Since we have $\exp(n(-h_\tau(\nu) - \varepsilon)) < \nu(I_n(x))$ for each $I_n(x) \in \mathcal{U}_n^N$, the cardinality of \mathcal{U}_n^N does not exceed $\exp(n(h_\tau(\nu) + \varepsilon))$. Furthermore, in view of (8), for each $I_n(x) \in \mathcal{U}_n^N$ we can construct a covering of $G_N \cap (I_n(x) \times \mathbb{R})$ by at most $\left\lceil \overline{C} \frac{\lambda^n(x)}{|I_n(x)|} \right\rceil$ cubes with edge length $|I_n(x)|$. Moreover, for $x \in D_N$ and $n \geq N$, this number is bounded as

$$\left\lceil \overline{C} \frac{\lambda^n(x)}{|I_n(x)|} \right\rceil \leq \overline{C} \exp \left(n \left(\int \log \tau' d\nu + \int \log \lambda d\nu + 2\varepsilon \right) \right) + 1.$$

Observe that the collection of all these cubes over $x \in [0, 1]$, which is denoted by $\tilde{\mathcal{U}}_n^N$, forms a covering of G_N . In addition, it satisfies

$$\begin{aligned} & \sum_{Q \in \tilde{\mathcal{U}}_n^N} \text{edge-length}(Q)^s \\ & \leq \sum_{I \in \mathcal{U}_n^N} \left(\overline{C} \exp \left(n \left(\int \log \tau' d\nu + \int \log \lambda d\nu + 2\varepsilon \right) \right) + 1 \right) \cdot |I|^s \\ & = \sum_{I \in \mathcal{U}_n^N} \left(\overline{C} \exp \left(n \left(\int \log \tau' d\nu + \int \log \lambda d\nu + \frac{s \log |I|}{n} + 2\varepsilon \right) \right) + |I|^s \right) \\ & \leq \sum_{I \in \mathcal{U}_n^N} \left(\overline{C} \cdot e^{n(\int \log \tau' d\nu + \int \log \lambda d\nu + s(-\int \log \tau' d\nu + \varepsilon) + 2\varepsilon)} + e^{sn(-\int \log \tau' d\nu + \varepsilon)} \right) \\ & \leq e^{n(h_\tau(\nu) + \varepsilon)} \cdot \left(\overline{C} \cdot e^{n(\int \log \tau' d\nu + \int \log \lambda d\nu + s(-\int \log \tau' d\nu + \varepsilon) + 2\varepsilon)} + e^{sn(-\int \log \tau' d\nu + \varepsilon)} \right) \\ & =: \overline{C} \cdot e^{n(h_\tau(\nu) + (1-s)\int \log \tau' d\nu + \int \log \lambda d\nu + (3+s)\varepsilon)} + e^{n(h_\tau(\nu) - s\int \log \tau' d\nu + (1+s)\varepsilon)} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

for each

$$s > f(\varepsilon) := \max \left\{ \frac{h_\tau(\nu) + \int \log \lambda d\nu + \int \log \tau' d\nu + 3\varepsilon}{\int \log \tau' d\nu - \varepsilon}, \frac{h_\tau(\nu) + \varepsilon}{\int \log \tau' d\nu - \varepsilon} \right\}.$$

Hence we have proved $\dim_H(G_N) \leq f(\varepsilon)$. Furthermore, in view of the σ -stability of the Hausdorff dimension, we have

$$\dim_H \left(\bigcup_N G_N \right) = \sup_N \dim_H(G_N) \leq f(\varepsilon).$$

As $\mu_\theta(\bigcup_N G_N) = 1$, we have $\dim_H(\mu_\theta) \leq \dim_H(\bigcup_N G_N)$ by definition. Thus,

$$\dim_H(\mu_\theta) \leq \lim_{\varepsilon \rightarrow 0} f(\varepsilon) = \max \left\{ 1 + \frac{h_\tau(\nu) + \int \log \lambda d\nu}{\int \log \tau' d\nu}, \frac{h_\tau(\nu)}{\int \log \tau' d\nu} \right\} = 1 + \frac{h_\tau(\nu) + \int \log \lambda d\nu}{\int \log \tau' d\nu}.$$

Now, we consider a very rough estimate as follows. Observe that for any $\beta \in (0, 1)$ and $C > 0$ we have

$$\underline{d}_{\mu_\theta}(x, W_\theta(x)) = \liminf_{n \rightarrow \infty} \frac{\log \nu \{v \in [0, 1] : |x - v| \leq \beta^n, |W_\theta(x) - W_\theta(v)| \leq C \beta^n\}}{\log \beta^n}$$

holds for ν -almost all x , which immediately follows from [Bar08, Proposition 2.1.4]. Given an arbitrary $\varepsilon \in (0, e^{\int \log \lambda d\nu})$, let $D_N \subseteq [0, 1]$, $N \in \mathbb{N}$ be the same as above. This time, we only use the first, third and last property of the definition of D_N , or more precisely, the fact that

$$I_n(x) \subseteq \left\{ v \in [0, 1] : |v - x| \leq (e^\varepsilon \tilde{\lambda})^n, |W_\theta(v) - W_\theta(x)| \leq \overline{C} (e^\varepsilon \tilde{\lambda})^n \right\}$$

holds for all $x \in D_N$, $n \geq N$, where $\tilde{\lambda} := e^{\int \log \lambda d\nu}$. By choosing $\beta = e^\varepsilon \tilde{\lambda}$ and $C = \overline{C}$ in the above equality, we have

$$\begin{aligned} \underline{d}_{\mu_\vartheta}(x, W_\vartheta(x)) &= \liminf_{n \rightarrow \infty} \frac{\log \nu \left\{ v \in [0, 1] : |x - v| \leq (e^\varepsilon \tilde{\lambda})^n, |W_\vartheta(v) - W_\vartheta(x)| \leq \overline{C} (e^\varepsilon \tilde{\lambda})^n \right\}}{\log(e^\varepsilon \tilde{\lambda})^n} \\ &\leq \liminf_{n \rightarrow \infty} \frac{\log \nu(I_n(x))}{\log(e^\varepsilon \tilde{\lambda})^n} \leq \frac{h_\tau(\nu)}{-\int \log \lambda d\nu - \varepsilon} \end{aligned}$$

for all $x \in D_N$ and $N \in \mathbb{N}$. In view of Lemma 3, letting $N \rightarrow \infty$ and $\varepsilon \rightarrow 0$ finishes the proof. \square

5.2 Lower bound

We assume that $\nu \in \mathcal{P}_\tau([0, 1])$ is a Gibbs measure. Moreover, we assume that g satisfies the critical point hypothesis. In addition, let d denote the Euclidean metric on $[0, 1]$, i.e. $d(x, v) := |x - v|$ for $x, v \in [0, 1]$. As τ has full branches and $\tau' > 0$, there is a small number $\delta_0 > 0$ such that for any $x, v \in [0, 1]$

$$\max\{d(x, v), d(\tau x, \tau v)\} \leq \delta_0 \quad \text{implies} \quad v \in I_1(x).$$

Clearly, δ_0 is an expansivity constant for τ . Given $n \in \mathbb{N}$, let $d_n(x, v) := \max\{d(\tau^j x, \tau^j v) : j = 0, \dots, n-1\}$ denote the n -the Bowen metric for $x, v \in [0, 1]$. In addition, let $B_{n,r}(x) := \{u \in [0, 1] : d_n(x, u) \leq r\}$ denote the ball with respect to d_n with centre $x \in [0, 1]$ and radius $r > 0$.

For $\varepsilon \in (0, 1)$, let $\mathcal{C}_\varepsilon \subseteq [0, 1]$ be the set of those x for which hold

- $B_{e^{-n(\int \log \tau' d\nu + \varepsilon)}}(x) \subseteq B_{n, \delta_0}(x) \subseteq B_{e^{-n(\int \log \tau' d\nu - \varepsilon)}}(x)$,
- $e^{n(\int \log \lambda d\nu - \varepsilon)} \leq \lambda^n(x) \leq e^{n(\int \log \lambda d\nu + \varepsilon)}$, and
- $r^{\dim_H(\nu) + \varepsilon} \leq \nu(B_r(x)) \leq r^{\dim_H(\nu) - \varepsilon}$

for all $n \in \mathbb{N}$ and $r > 0$.

Lemma 19. *We have $\lim_{\varepsilon \rightarrow 0} \nu(\mathcal{C}_\varepsilon) = 1$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Further, let $x \in [0, 1]$ and $n \geq 2$. Firstly, if $v \in [0, 1]$ satisfies $d_n(x, v) \leq \delta_0$, then $v \in I_{n-2}(x)$ due to the choice of δ_0 . Thus we have $d(x, v) \leq |I_{n-2}(x)| \leq D/(\tau^{n-2})'(x)$ by Lemma 12. In consequence, we have

$$B_{n, \delta_0}(x) \subseteq B_{R_n(x)}(x), \quad \text{where} \quad R_n(x) := \frac{D}{(\tau^{n-2})'(x)}.$$

Next, suppose that $v \in [0, 1]$ satisfies $d_n(x, v) > \delta_0$, i.e. that there is some $0 \leq j \leq n-1$ such that $d(\tau^j x, \tau^j v) > \delta_0$. Here, we distinguish two cases. If $v \notin I_n(x)$, then we have $d(x, v) \geq \text{dist}(\partial I_n(x), x)$. And, otherwise we have

$$(\tau^n)'(\tilde{v}) \cdot d(x, v) \geq (\tau^j)'(\tilde{v}) \cdot d(x, v) = d(\tau^j x, \tau^j v) > \delta$$

for some $\tilde{v} \in I_n(x)$, so that follows $d(x, v) > D^{-1}\delta/(\tau^n)'(x)$. In consequence, we have

$$B_{r_n(x)}(x) \subseteq B_{n, \delta_0}(x), \quad \text{where} \quad r_n(x) := \min \left\{ \text{dist}(\partial I_n(x), x), \frac{D^{-1}\delta_0}{(\tau^n)'(x)} \right\}.$$

Now, let $C \subseteq [0, 1]$ be the set of those x for which hold

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log r_n(x)}{n} &= \lim_{n \rightarrow \infty} \frac{\log R_n(x)}{n} = -\int \log \tau' d\nu, \\ \lim_{n \rightarrow \infty} \frac{\log \lambda^n(x)}{n} &= \int \log \lambda d\nu \quad \text{and} \quad \lim_{r \searrow 0} \frac{\log \nu(B_r(x))}{\log r} = \dim_H \nu. \end{aligned}$$

Observe that $\nu(C) = 1$. Indeed, the first condition follows from Birkhoff's theorem together with Lemmas 12 and 14, while the second and the last one are due to Birkhoff's theorem and Lemma 15, respectively. In view of $C \subseteq \bigcup_{\varepsilon > 0} \mathcal{C}_\varepsilon$, the proof is finished. \square

We consider the restriction $\nu_\varepsilon := \nu(\mathcal{C}_\varepsilon \cap \cdot)$ and its lift

$$\mu_{\boldsymbol{\vartheta}, \varepsilon} := \nu \{x \in [0, 1] : (x, W_{\boldsymbol{\vartheta}}(x)) \in \cdot\}.$$

Recall that s -energy of a Borel measure μ on a metric space $(\mathcal{E}, d_\mathcal{E})$ is defined by

$$I_s(\mu) = \iint \frac{d\mu(x)d\mu(v)}{d_\mathcal{E}(x, v)^s}.$$

Furthermore, $I_s(\mu) < \infty$ implies that $\underline{d}_\mu \geq s$ holds μ -almost surely (see e.g. [Fal05, Section 4.3]). Observe that

$$\begin{aligned} \int I_s(\mu_{\boldsymbol{\vartheta}, \varepsilon}) d\boldsymbol{\vartheta} &= \iiint \frac{d\nu_\varepsilon(x) d\nu_\varepsilon(v) d\boldsymbol{\vartheta}}{(d(x, v)^2 + (W_{\boldsymbol{\vartheta}}(x) - W_{\boldsymbol{\vartheta}}(v))^2)^{s/2}} \\ &\leq \frac{1}{\delta_0^s} + \iiint_{\{(x, v) \in \Delta_{\delta_0}\}} \frac{d\nu_\varepsilon(x) d\nu_\varepsilon(v) d\boldsymbol{\vartheta}}{(d(x, v)^2 + (W_{\boldsymbol{\vartheta}}(x) - W_{\boldsymbol{\vartheta}}(v))^2)^{s/2}} \\ &=: \frac{1}{\delta_0^s} + E_{\varepsilon, s}, \end{aligned} \tag{9}$$

where $\Delta_{\delta_0} := \{(x, v) \in [0, 1]^2 : d(x, v) \leq \delta_0\}$ is the δ_0 -diagonal set.

Lemma 20. *If $E_{\varepsilon, s} < \infty$ for some $s > 0$, then we have almost surely that*

$$\underline{d}_{\mu_{\boldsymbol{\vartheta}}} (x, W_{\boldsymbol{\vartheta}}(x)) \geq s$$

holds for ν -a.a. $x \in \mathcal{C}_\varepsilon$.

Proof. If $E_{\varepsilon, s} < \infty$, we have $I_s(\mu_{\boldsymbol{\vartheta}, \varepsilon}) < \infty$ by (9) for almost all $\boldsymbol{\vartheta}$. In the following, let $\boldsymbol{\vartheta}$ be such a parameter. Now, the bounded s -energy implies that $\underline{d}_{\mu_{\boldsymbol{\vartheta}, \varepsilon}}(x, y) \geq s$ holds for $\mu_{\boldsymbol{\vartheta}, \varepsilon}$ -almost all $(x, y) \in [0, 1] \times \mathbb{R}$. This means by definition that $\underline{d}_{\mu_{\boldsymbol{\vartheta}, \varepsilon}}(x, W_{\boldsymbol{\vartheta}}(x)) \geq s$ holds for ν -almost all $x \in \mathcal{C}_\varepsilon$. On the other hand, by Borel density theorem we have

$$\lim_{r \rightarrow 0} \frac{\mu_{\boldsymbol{\vartheta}, \varepsilon}(B_r(x, y))}{\mu_{\boldsymbol{\vartheta}}(B_r(x, y))} = \lim_{r \rightarrow 0} \frac{\mu_{\boldsymbol{\vartheta}}(B_r(x, y) \cap (\mathcal{C}_\varepsilon \times \mathbb{R}))}{\mu_{\boldsymbol{\vartheta}}(B_r(x, y))} = 1$$

for $\mu_{\boldsymbol{\vartheta}}$ -almost all $(x, y) \in \mathcal{C}_\varepsilon \times \mathbb{R}$. Thus, we have $\underline{d}_{\mu_{\boldsymbol{\vartheta}}}(x, W_{\boldsymbol{\vartheta}}(x)) = \underline{d}_{\mu_{\boldsymbol{\vartheta}, \varepsilon}}(x, W_{\boldsymbol{\vartheta}}(x)) \geq s$ for ν -almost all $x \in \mathcal{C}_\varepsilon$. \square

Given $x, v \in [0, 1]$, let $h_{x, v}$ be the density function of the random variable $\boldsymbol{\vartheta} \mapsto W_{\boldsymbol{\vartheta}}(x) - W_{\boldsymbol{\vartheta}}(v)$ with respect to the Lebesgue measure on \mathbb{R} . The next key lemma is due to [MW12], which in particular says that the densities exist.

Lemma 21 (Key lemma). *There is a constant $C_h > 0$ such that*

$$\|h_{x, v}\|_\infty \leq \frac{C_h}{\lambda^n(x)}$$

for all $v \notin B_{n, \delta_0}(x) \setminus B_{n+1, \delta_0}(x)$, $n \in \mathbb{N}$ and $x \in [0, 1]$.

Proof. This is a special case of [MW12, Lemma 4.2], whose proof can almost literally translated to this case. Some remarks:

- The points z, x, y in their proof correspond to x, x, v in our notation.
- The restriction to $B_{n, 2\delta_0}(z) \times B_{n, 2\delta_0}(z)$ in the definition of $X_n^r(z)$ in their proof was not used for this part, i.e. $X_n^r(z)$ corresponds to our $B_{n, \delta_0}(x) \setminus B_{n+1, \delta_0}(x)$.
- We allow here possible non-differentiability of τ and λ at the end points of Markov partition. This does not matter, while $g \in C^\infty(\mathbb{R}/\mathbb{Z})$ is a crucial assumption here.

□

Lemma 22. Suppose that $-\int \log \lambda d\nu < h_\tau(\nu)$. Then we have almost surely that

$$\underline{d}_{\mu_\vartheta} \geq \dim_H \nu + 1 + \frac{\int \log \lambda d\nu}{\int \log \tau' d\nu}$$

holds μ_ϑ -almost surely.

Proof. For $s > 1$, by Lemma 21 and the substitution formula of integral, we have

$$\begin{aligned} \int \frac{d\vartheta}{(d(x, v)^2 + (W_\vartheta(x) - W_\vartheta(v))^2)^{s/2}} &= \int \frac{h_{x,v}(z) dz}{(d(x, v)^2 + z^2)^{s/2}} \\ &= d(x, v)^{1-s} \int \frac{h_{x,v}(d(x, v)t) dt}{(1+t^2)^{s/2}} \\ &\leq d(x, v)^{1-s} \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{s/2}} \|h_{x,v}\|_\infty \\ &= d(x, v)^{1-s} K_s \|h_{x,v}\|_\infty \leq \frac{K_s C_h}{\lambda^n(x) d(x, v)^{s-1}}, \end{aligned}$$

where the constant $K_s := \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{s/2}} < \infty$ as $s > 1$. Let $\varepsilon > 0$ be small enough and let $x \in \mathcal{C}_\varepsilon$. By Fubini's lemma we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \iint_{B_{n,\delta}(x) \setminus B_{n+1,\delta}(x)} \frac{d\nu_\vartheta(v)}{(d(x, v)^2 + (W_\vartheta(x) - W_\vartheta(v))^2)^{s/2}} d\vartheta \\ &\leq K_s C_h \sum_{n=1}^{\infty} \frac{1}{\lambda^n(x)} \int_{B_{n,\delta}(x) \setminus B_{n+1,\delta}(x)} \frac{d\nu_\varepsilon(v)}{d(x, v)^{s-1}} \\ &\leq K_s C_h \sum_{n=1}^{\infty} \frac{1}{\lambda^n(x)} \int_{B_{e^{-n}(\int \log \tau' d\nu - \varepsilon)}(x) \setminus B_{e^{-n}(\int \log \tau' d\nu + \varepsilon)}(x)} \frac{d\nu_\varepsilon(v)}{d(x, v)^{s-1}} \\ &\leq K_s C_h \sum_{n=1}^{\infty} \frac{1}{\lambda^n(x)} \frac{\nu(B_{e^{-n}(\int \log \tau' d\nu + \varepsilon)}(x))}{e^{-n(s-1)(\int \log \tau' d\nu - \varepsilon)}} \\ &\leq K_s C_h \sum_{n=1}^{\infty} \frac{\left(e^{-n(\int \log \tau' d\nu - \varepsilon)}\right)^{\dim_H \nu - \varepsilon}}{e^{n(\int \log \lambda d\nu - \varepsilon) - n(s-1)(\int \log \tau' d\nu + \varepsilon)}} \\ &= K_s C_h \sum_{n=1}^{\infty} e^{n(-(\int \log \tau' d\nu - \varepsilon)(\dim_H \nu - \varepsilon) - \int \log \lambda d\nu + \varepsilon + (s-1)(\int \log \tau' d\nu + \varepsilon))} < \infty \end{aligned}$$

for all $s \in (1, S(\varepsilon))$, where

$$S(\varepsilon) := 1 + \frac{(\int \log \tau' d\nu - \varepsilon)(\dim_H \nu - \varepsilon) + \int \log \lambda d\nu - \varepsilon}{\int \log \tau' d\nu + \varepsilon}.$$

By assumption, we have that $\lim_{\varepsilon \rightarrow 0} S(\varepsilon) = \dim_H \nu + 1 + \frac{\int \log \lambda d\nu}{\int \log \tau' d\nu} > 1$. Furthermore, by integration over $x \in \mathcal{C}_\varepsilon$, we have $E_{\varepsilon,s} < \infty$ for all $s \in (1, S(\varepsilon))$. Thus, the claim follows by lemmas 19 and 20.

□

Lemma 23. Suppose that $-\int \log \lambda d\nu \geq h_\tau(\nu)$. Then we have almost surely that

$$\underline{d}_{\mu_\vartheta} \geq \frac{h_\tau(\nu)}{\int \log \tau' d\nu}$$

holds μ_ϑ -almost surely.

Proof. For $s < s' < 1$, by Jensen's inequality and Lemma 21, we have

$$\begin{aligned}
& \int \frac{d\boldsymbol{\vartheta}}{(d(x, v)^2 + (W_{\boldsymbol{\vartheta}}(x) - W_{\boldsymbol{\vartheta}}(v))^2)^{s/2}} \\
&= \int \frac{h_{x,v}(z) dz}{(d(x, v)^2 + z^2)^{s/2}} \\
&\leq \left(\int \frac{h_{x,v}(z) dz}{(d(x, v)^2 + z^2)^{s/(2s')}} \right)^{s'} \\
&\leq \left(\int_{-\infty}^{\infty} \frac{dz}{(d(x, v)^2 + z^2)^{s/(2s')}} \right)^{s'} \|h_{x,v}\|_{\infty}^{s'} \\
&= d(x, v)^{s'-s} \left(\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{s/(2s')}} \right)^{s'} \|h_{x,v}\|_{\infty}^{s'} \\
&= K_{s'/s} d(x, v)^{s'-s} \|h_{x,v}\|_{\infty}^{s'} \leq \frac{C_h^{s'} K_{s'/s}}{(\lambda^n(x))^{s'}},
\end{aligned}$$

where K_t is the same constant as that in proof of Lemma 22 and bounded for $t = s'/s > 1$. Let $\varepsilon > 0$ be small enough and let $x \in \mathcal{C}_{\varepsilon}$. By Fubini's lemma we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \iint_{B_{n,\delta}(x) \setminus B_{n+1,\delta}(x)} \frac{d\nu_{\boldsymbol{\vartheta}}(v)}{(d(x, v)^2 + (W_{\boldsymbol{\vartheta}}(x) - W_{\boldsymbol{\vartheta}}(v))^2)^{s/2}} d\boldsymbol{\vartheta} \\
&\leq C_h^{s'} K_{s'/s} \sum_{n=1}^{\infty} \frac{\nu(B_{n,\delta}(x))}{(\lambda^n(x))^{s'}} \\
&\leq C_h^{s'} K_{s'/s} \sum_{n=1}^{\infty} \frac{\nu(B_{e^{-n(\int \log \tau' d\nu - \varepsilon)}}(x))}{(\lambda^n(x))^{s'}} \\
&\leq C_h^{s'} K_{s'/s} \sum_{n=1}^{\infty} e^{-n(\int \log \tau' d\nu - \varepsilon)(\dim_H \nu - \varepsilon)} e^{-s'n(\int \log \lambda d\nu - \varepsilon)} < \infty
\end{aligned}$$

for all $s \in (0, 1)$ and $s' \in (\max\{s, \tilde{S}(\varepsilon)\}, 1)$, where

$$\tilde{S}(\varepsilon) := \frac{(\int \log \tau' d\nu - \varepsilon)(\dim_H \nu - \varepsilon)}{-\int \log \lambda d\nu + \varepsilon}.$$

Observe that $\tilde{S}(\varepsilon) < 1$ for $\varepsilon > 0$ so that the following approximation works, even in case $\tilde{S}(0) = 1$. Since $\lim_{\varepsilon \rightarrow 0} \tilde{S}(\varepsilon) = \frac{h_{\tau}(\nu)}{\int \log \tau' d\nu}$, the claim follows by lemmas 19 and 20. \square

Conclusion of proof of Theorem 1. From Lemmas 22 and 23 follow the lower estimate, i.e. it is almost sure that

$$\underline{d}_{\mu_{\boldsymbol{\vartheta}}} \geq \min \left\{ \dim_H \nu + 1 + \frac{\int \log \lambda d\nu}{\int \log \tau' d\nu}, \frac{h_{\tau}(\nu)}{\int \log \tau' d\nu} \right\}$$

holds $\mu_{\boldsymbol{\vartheta}}$ -almost surely. \square

6 Dimension of the graph

Recall that s_1 and s_2 are the numbers defined in (6).

Lemma 24. *We have $s_1, s_2 \in (\dim_H \mathcal{J}, 2)$.*

Proof. This follows from the facts $|\tau'\lambda| > 1$, $P(0) > 0$ and $P(-(\dim_H \mathcal{J}) \cdot \log |\tau'|) = 0$. \square

Basically, all canonical upper bounds for several types of the dimension of \mathcal{GW}_ϑ follow from the next key lemma which originates in [Bed89a]. For technical reasons we postpone its proof.

Lemma 25. *There is a constant $\overline{C} > 0$ such that*

$$\sup_{v \in \mathcal{J} \cap I_n(x)} |W_\vartheta(x) - W_\vartheta(v)| \leq \overline{C} \lambda^n(x)$$

for all $x \in [0, 1]$, $n \in \mathbb{N}$ and $\vartheta \in [0, 1]^{\mathbb{N}_0}$.

Proof. See Lemma 29. \square

6.1 Upper bounds for $\overline{\dim}_B \mathcal{GW}_\vartheta$ and $\dim_H \mathcal{GW}_\vartheta$

Lemma 26. *We have $\overline{\dim}_B \mathcal{GW}_\vartheta \leq s_1$ for all $\vartheta \in [0, 1]^{\mathbb{N}_0}$.*

Proof. Given ϑ , let N_r be the least number of squares with side length r that are needed to cover \mathcal{GW}_ϑ . Recall that $\overline{\dim}_B \mathcal{GW}_\vartheta = \limsup_{r \rightarrow 0} \frac{\log N_r}{\log r}$. For $r > 0$, let $\mathfrak{U}_r := \{I_{n_r(x)}(x) : x \in \mathcal{J}\}$, where $n_r(x) := \min\{n \in \mathbb{N} : |I_n(x)| \leq r\}$. Observe that $\mathcal{J} \subseteq \bigcup_{I \in \mathfrak{U}_r} \overline{I}$ and $\delta_0 r \leq |I| \leq r$ for all $I \in \mathfrak{U}_r$, where $\delta_0 > 0$ is the constant from Lemma 12. In view of Lemma 25, for each $I_n(x) \in \mathfrak{U}_r$, the part $(\mathcal{GW}_\vartheta) \cap (\overline{I_n(x)} \times \mathbb{R})$ of the graph can be covered by $\lceil \overline{C} \lambda^n(x) / |I_n(x)| \rceil$ squares with side length r . Now, let $\nu_1 \in \mathcal{P}_\tau(\mathcal{J})$ be the equilibrium state for the topological pressure of the definition of s_1 in 6. By Lemmas 2 and 12, there is a constant $C > 0$ such that

$$C^{-1} \leq \frac{\nu_1(I_n(x))}{|I_n(x)|^{s_1-1} \lambda^n(x)} \leq C$$

holds for all $x \in \mathcal{J}$ and $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} N_r &\leq \sum_{I_n(x) \in \mathfrak{U}_r} \left\lceil \overline{C} \frac{\lambda^n(x)}{|I_n(x)|} \right\rceil \leq \sum_{I \in \mathfrak{U}_r} (\overline{C} + 1) C \nu_1(I) |I|^{-s_1} \\ &\leq (\overline{C} + 1) C (\delta_0 r)^{-s_1} \sum_{I \in \mathfrak{U}_r} \nu_1(I) = (\overline{C} + 1) C \delta_0^{s_1} r^{-s_1} \end{aligned}$$

for all $r > 0$. This finishes the proof. \square

Lemma 27. *We have $\dim_H \mathcal{GW}_\vartheta \leq \min\{s_1, s_2\}$ for all $\vartheta \in [0, 1]^{\mathbb{N}_0}$.*

Proof. As $\dim_H \mathcal{GW}_\vartheta \leq \overline{\dim}_B \mathcal{GW}_\vartheta \leq s_1$ by Lemma 26, we will now show that $\dim_H \mathcal{GW}_\vartheta \leq s_2$. To this end, observe that by Lemmas 17 and 25 we have

$$\text{hol}_{W_\vartheta}(x) = \liminf_{n \rightarrow \infty} \inf_{u \in \mathcal{J} \cap I_n(x)} \frac{\log |W_\vartheta(x) - W_\vartheta(u)|}{\log |I_n(x)|} \geq \liminf_{n \rightarrow \infty} \frac{-\log \lambda^n(x)}{\log |I_n(x)|} =: \tilde{h}(x) \quad (10)$$

for all $x \in \mathcal{J} \setminus \mathcal{N}$.

Now, let $E_\alpha^< := \{x \in \mathcal{J} : \tilde{h}(x) < \alpha\}$ for $\alpha > 0$, and we claim that

$$\dim_H E_\alpha^< \leq s_2 \alpha$$

for all $\alpha > 0$. Indeed, given $r > 0$, we can define

$$n_r(x) := \min\{n \in \mathbb{N} : |I_n(x)| \leq r \text{ and } |I_n(x)|^\alpha \leq \lambda^n(x)\}$$

for each $x \in E_\alpha^<$. Then, let $\mathfrak{U}_r := \{I_{n_r(x)}(x) : x \in E_\alpha^<\}$ so that $E_\alpha^< \subseteq \bigcup_{I \in \mathfrak{U}_r} \bar{I}$ and $|I| \leq r$ for all $I \in \mathfrak{U}_r$. Furthermore, let $\nu_2 \in \mathcal{P}_\tau(\mathcal{J})$ be the equilibrium state for the topological pressure of the definition of s_2 in (6). By Lemma 2 there is a constant $C > 0$ such that

$$C^{-1} \leq \frac{\nu_2(I_n(x))}{(\lambda(x))^{s_2}} \leq C$$

holds for all $x \in \mathcal{J}$ and $n \in \mathbb{N}$. In particular, for each $I_n(x) \in \mathfrak{U}_r$ we have

$$|I_n(x)|^{s_2\alpha} \leq (\lambda^n(x))^{s_2} \leq C \nu_2(I_n(x)),$$

where the first inequality is due to the construction of \mathfrak{U}_r . Thus, for any $d > s_2\alpha$, the d -Housdorff measure of $E_\alpha^<$ is bounded as

$$\begin{aligned} \mathcal{H}^d(E_\alpha^<) &\leq \sum_{I \in \mathfrak{U}_r} |I|^d = \sum_{I_n(x) \in \mathfrak{U}_r} |I_n(x)|^{d-s_2\alpha} |I_n(x)|^{s_2\alpha} \\ &\leq C r^{d-s_2\alpha} \sum_{I \in \mathfrak{U}_r} \nu_2(I) \leq C r^{d-s_2\alpha}. \end{aligned}$$

By letting $r \rightarrow 0$, we obtain $\mathcal{H}^d(E_\alpha^<) = 0$, i.e. $\dim_H E_\alpha^< \leq d$. Finally, letting $d \rightarrow s_2\alpha$ finishes the proof of this claim.

The rest of the proof is based on the general fact that

$$\dim_H \{(x, W_\vartheta(x)) : x \in E_\alpha^< \text{ and } \text{hol}_{W_\vartheta}(x) \geq \beta\} \leq \frac{\dim_H E_\alpha^<}{\beta}$$

holds for any $\alpha, \beta > 0$, see [Jin11, Theorem 1]. Given $N \in \mathbb{N}$, let $t_i := \alpha_{\min} + \frac{i}{N}(\alpha_{\max} - \alpha_{\min})$ for $i = 0, \dots, N$. Observe that for each $x \in \mathcal{J} \setminus \mathcal{N}$ there is some $i \in \{0, \dots, N-1\}$ such that $t_i \leq \tilde{h}(x) < t_{i+1}$, which in turn implies by (10) that $\text{hol}_{W_\vartheta}(x) \geq \tilde{h}(x) \geq t_i$. Thus, we have

$$\mathcal{J} \setminus \mathcal{N} = \bigcup_{i=0}^{N-1} \{x \in \mathcal{J} \setminus \mathcal{N} : \tilde{h}(x) \leq t_{i+1} \text{ and } \text{hol}_{W_\vartheta}(x) \geq t_i\}.$$

for any $\alpha, \beta > 0$. As the countable set \mathcal{N} is negligible under the Hausdorff dimension, we have

$$\begin{aligned} \dim_H \mathcal{GW}_\vartheta &= \max_{i=0, \dots, N-1} \dim_H \left\{ (x, W_\vartheta(x)) : x \in E_{t_{i+1}}^< \text{ and } \text{hol}_{W_\vartheta}(x) \geq t_i \right\} \\ &\leq \max_{i=0, \dots, N-1} \frac{\dim_H E_{t_{i+1}}^<}{t_i} \\ &\leq \max_{i=0, \dots, N-1} \frac{s_2 t_{i+1}}{t_i} \leq s_2 \left(1 + \frac{1}{\alpha_{\min} N} \right), \end{aligned}$$

which finishes the proof by $N \rightarrow \infty$. □

6.2 Lower bound

Lemma 28. *Assume that there is a sequence $(c_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $\lim_{n \rightarrow \infty} \frac{\log c_n}{n} = 0$ such that*

$$c_n \lambda^n(x) \leq \sup_{v \in \mathcal{J} \cap I_n(x)} |W_\vartheta(v) - W_\vartheta(x)| \quad (11)$$

for all $x \in \mathcal{J}$ and $n \in \mathbb{N}$. Then we have $\underline{\dim}_B \mathcal{GW}_\vartheta \geq s_1$.

Proof. Let $\nu_1 \in \mathcal{P}_\tau(\mathcal{J})$ and $C > 0$ be the same as in the proof of Lemma 26, i.e. we have

$$C^{-1} \leq \frac{\nu_1(I_n(x))}{|I_n(x)|^{s_1-1} \lambda^n(x)} \leq C$$

for all $x \in \mathcal{J}$ and $n \in \mathbb{N}$. In addition, let N_r be the minimal number of the boxes with side length $r > 0$ which are needed to cover \mathcal{GW}_ϑ so that

$$\underline{\dim}_B \mathcal{GW}_\vartheta = \liminf_{r \rightarrow 0} \frac{\log N_r}{-\log r}.$$

Now, let $D_n \subseteq [0, 1]$ denote the set of those $x \in \mathcal{J}$ which satisfy: For all $n \geq N$ hold all of

- $\left| \frac{\log \lambda^n(x)}{n} - \int \log \lambda d\nu_1 \right| < \varepsilon,$
- $\left| \frac{-\log \nu_1(I_n(x))}{n} - h_\tau(\nu_1) \right| < \varepsilon$ and
- $\left| \frac{-\log |I_n(x)|}{n} - \int \log |\tau'| d\nu_1 \right| < \varepsilon.$

As $\lim_{N \rightarrow \infty} \nu_1(D_N) = 1$, we can choose a number $N_0 \in \mathbb{N}$ such that $\nu_1(D_{N_0}) > 0$. Let $\mathfrak{U}_n := \{I_n(x) : x \in D_{N_0}\}$ for $n \geq N_0$. Since

$$C \geq \frac{\nu_1(I_n(x))}{|I_n(x)|^{s_1-1} \lambda^n(x)} \geq \nu(I_n(x)) e^{n(-(1-s_1) \int \log |\tau'| d\nu_1 - \log \lambda d\nu_1 - s_1 \varepsilon)}$$

holds for each $I_n(x) \in \mathfrak{U}_n$, we have

$$\begin{aligned} \#\mathfrak{U}_n &\geq C^{-1} \sum_{I \in \mathfrak{U}_n} e^{n(-(1-s_1) \int \log |\tau'| d\nu_1 - \log \lambda d\nu_1 - s_1 \varepsilon)} \nu_1(I) \\ &\geq C^{-1} \nu_1(D_{N_0}) e^{n(-(1-s_1) \int \log |\tau'| d\nu_1 - \log \lambda d\nu_1 - s_1 \varepsilon)} \end{aligned}$$

for all $n \in \mathbb{N}$. On the other hand, in view of the assumption (11), for each $I \in \mathfrak{U}_n$, the height of the part $\mathcal{GW}_\vartheta \cap (I \times \mathbb{R})$ of the graph is at least $e^{n(\int \log \lambda d\nu_1 - \varepsilon) + \log c_n}$. In addition, W_ϑ is continuous. Let $r_n := e^{n(-\int \log |\tau'| d\nu_1 - \varepsilon)}$. Since $|I| \leq r_n$ for all $n \in \mathfrak{U}_n$, we have

$$N_{r_n} \geq \#\mathfrak{U}_n \frac{e^{n(\int \log \lambda d\nu_1 - \varepsilon) - \log c_n}}{2r_n} \geq (2C)^{-1} \nu_1(D_{N_0}) e^{n(s_1 \int \log |\tau'| d\nu_1 - (s_1+2)\varepsilon) - \log c_n}.$$

Consequently, we have

$$\liminf_{r \rightarrow 0} \frac{\log N_r}{-\log r} = \liminf_{n \rightarrow \infty} \frac{\log N_{r_n}}{-\log r_n} \geq \frac{s_1 \int \log |\tau'| d\nu_1 - (s_1+2)\varepsilon}{\int \log |\tau'| d\nu_1 + \varepsilon},$$

where the above equality is due to the monotonicity of the sequences $(r_n)_{n \in \mathbb{N}}$ and $(N_r)_{r>0}$, and the fact $\lim_{n \rightarrow \infty} \frac{\log r_n}{\log r_{n+1}} = 1$. Letting $\varepsilon \rightarrow 0$ finishes the proof. \square

Lemma 29. *The followings are true.*

- *There is a constant $\overline{C} > 0$ such that*

$$\sup_{v \in \mathcal{J} \cap I_n(x)} |W_\vartheta(x) - W_\vartheta(v)| \leq \overline{C} \lambda^n(x)$$

for all $x \in [0, 1]$, $n \in \mathbb{N}$ and $\vartheta \in [0, 1]^{\mathbb{N}_0}$.

- *If W is non-degenerate, then there is a constant $c > 0$ such that*

$$c \lambda^n(x) \leq \sup_{v \in \mathcal{J} \cap I_n(x)} |W(x) - W(v)|$$

for all $x \in [0, 1]$ and $n \in \mathbb{N}$.

- If $W_{\boldsymbol{\vartheta}}$ is non-degenerate for almost all $\boldsymbol{\vartheta}$, then there is a measurable function $c : [0, 1]^{\mathbb{N}_0} \rightarrow [0, \infty)$ with $\lim_{n \rightarrow \infty} \frac{\log c(\sigma^n \boldsymbol{\vartheta})}{n} = 0$ a.s. such that

$$c(\sigma^n \boldsymbol{\vartheta}) \lambda^n(x) \leq \sup_{v \in \mathcal{J} \cap I_n(x)} |W_{\boldsymbol{\vartheta}}(x) - W_{\boldsymbol{\vartheta}}(v)|$$

for all $x \in [0, 1]$, $n \in \mathbb{N}$ and $\boldsymbol{\vartheta} \in [0, 1]^{\mathbb{N}_0}$, where $\sigma : [0, 1]^{\mathbb{N}_0} \rightarrow [0, 1]^{\mathbb{N}_0}$ is the left shift operator.

The following is an immediate consequence of Lemma 28 and the last claim of Lemma 29.

Corollary 30. *If $W_{\boldsymbol{\vartheta}}$ is non-degenerate for almost all $\boldsymbol{\vartheta}$, then we have $\underline{\dim}_B \mathcal{GW}_{\boldsymbol{\vartheta}} \geq s_1$ for almost all $\boldsymbol{\vartheta}$.*

To prove Lemma 29, we need to prepare auxiliary lemmas. Except for Lemma 33 on the random lower bound, these are slight modifications of key arguments in [Bed89a].

Here, the random elements $\boldsymbol{\vartheta}$ need to be handled as an additional variable. One of our goals is to apply an ergodic theorem to the shift dynamics on $([0, 1]^{\mathbb{N}_0}, m^{\mathbb{N}_0})$, where $m^{\mathbb{N}_0}$ is the infinite product of the Lebesgue measure m on $[0, 1]$.

For each $(\vartheta, i) \in [0, 1] \times \{0, \dots, \ell - 1\}$, we define the contracting map $F_{\vartheta, i} : [0, 1] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}$ by

$$F_{\vartheta, i}(x, y) = (\rho_i(x), \lambda(\rho_i(x))y + g(\rho_i(x) + \vartheta)).$$

Observe that

$$F_{\vartheta_0, \kappa(x)}(\tau x, W_{\sigma \boldsymbol{\vartheta}}(\tau x)) = (x, W_{\boldsymbol{\vartheta}}(x)) \quad (12)$$

for all $x \in \mathcal{J}$ and $\boldsymbol{\vartheta} \in [0, 1]^{\mathbb{N}_0}$, where $[\boldsymbol{\vartheta}]_n := (\vartheta_0, \vartheta_1, \dots, \vartheta_{n-1})$. In addition, the derivative matrix of $F_{\vartheta, i}$ can be calculated as

$$\begin{aligned} DF_{\vartheta, i}(x, y) &= \begin{bmatrix} \rho'_i(x) & 0 \\ (y \cdot \lambda' + g'(\cdot + \vartheta))(\rho_i(x)) \cdot \rho'_i(x) & \lambda(\rho_i(x)) \end{bmatrix} \\ &=: \begin{bmatrix} a_i(x) & 0 \\ b_i(\vartheta, x, y) & c_i(x) \end{bmatrix} \end{aligned}$$

for $(x, y) \in [0, 1] \times \mathbb{R}$. As usual, the iterated maps are denoted by

$$\rho_i := \rho_{i_{n-1}} \circ \dots \circ \rho_{i_1} \circ \rho_{i_0} \quad \text{and} \quad F_{\boldsymbol{\vartheta}, \mathbf{i}} := F_{\vartheta_{n-1}, i_{n-1}} \circ \dots \circ F_{\vartheta_1, i_1} \circ F_{\vartheta_0, i_0}$$

for $(\boldsymbol{\vartheta}, \mathbf{i}) \in [0, 1]^n \times \{0, \dots, \ell - 1\}^n$. By (12), we have

$$F_{[\boldsymbol{\vartheta}]_n, [\mathbf{i}]_n}(\tau^n x, W_{\sigma^n \boldsymbol{\vartheta}}(\tau^n x)) = (x, W_{\boldsymbol{\vartheta}}(x)) \quad (13)$$

for all $x \in \mathcal{J}$, $\boldsymbol{\vartheta} \in [0, 1]^{\mathbb{N}_0}$ and $n \in \mathbb{N}$.

In the following lemmas, a continuous map $C : I \rightarrow [0, 1] \times \mathbb{R}$ shall be called a curve, where $I \subseteq [0, 1]$ is a subinterval. We use the expression $C(t) = (C_1(t), C_2(t))$. Given a curve $C : I \rightarrow [0, 1] \times \mathbb{R}$, we define its width and height with respect to \mathcal{J} , respectively, as

$$|C|_{\mathcal{J}, W} := |I| \quad \text{and} \quad |C|_{\mathcal{J}, H} := \sup_{t_1, t_2 \in \mathcal{J} \cap I} |C_2(t_1) - C_2(t_2)|.$$

Lemma 31. *Let $M > 0$ and $I \subseteq [0, 1]$ be a subinterval. For any curve $C : I \rightarrow [0, 1] \times [-M, M]$, we have*

$$\inf_{t \in I} c_i(C_1(t)) - b \cdot |C|_{\mathcal{J}, W} \leq |F_{\vartheta, i} \circ C|_{\mathcal{J}, H} \leq \sup_{t \in I} c_i(C_1(t)) + b \cdot |C|_{\mathcal{J}, W},$$

where $b := \max \left\{ b_i(\vartheta, x, y) : \begin{array}{l} i \in \{0, \dots, \ell - 1\}, \vartheta \in [0, 1], \\ (x, y) \in [0, 1] \times [-M, M] \end{array} \right\}.$

Proof. In view of Stone-Weierstrass theorem, we may assume that C is a C^1 -function on I . Furthermore, we may also assume that $C'_1 \geq 0$ by reparametrisation if necessary. Since we have

$$(F_{\vartheta,i} \circ C)'(t) = DF_{\vartheta,i}(C(t)) \cdot \begin{bmatrix} C'_1(t) \\ C'_2(t) \end{bmatrix} = \begin{bmatrix} a_i(C_1(t)) C'_1(t) \\ b_i(\vartheta, C(t)) C'_1(t) + c_i(C_1(t)) C'_2(t) \end{bmatrix}$$

for all $t \in [0, 1]$, we have

$$\begin{aligned} |F_{\vartheta,i} \circ C|_{\mathcal{J},H} &\geq \int_{t_1}^{t_2} b_i(\vartheta, C(t)) C'_1(t) + c_i(C_1(t)) C'_2(t) dt \\ &\geq \inf_{t \in [t_1, t_2]} c_i(C_1(t)) \int_{t_1}^{t_2} C'_2(t) dt - b \int_{t_1}^{t_2} |C'_1(t)| dt \\ &\geq \inf_{t \in I} c_i(C_1(t)) \cdot (C_2(t_1) - C_2(t_2)) - b |C|_{\mathcal{J},W} \end{aligned}$$

for all $t_1, t_2 \in \mathcal{J} \cap I$. The first inequality follows from this. On the other hand, there are specific $t_1, t_2 \in \mathcal{J} \cap I$ such that

$$\begin{aligned} |F_{\vartheta,i} \circ C|_{\mathcal{J},H} &= \int_{t_1}^{t_2} b_i(\vartheta, C(t)) C'_1(t) + c_i(C_1(t)) C'_2(t) dt \\ &\leq b |C|_{\mathcal{J},W} + \int_{t_1}^{t_2} c_i(C_1(t)) C'_2(t) dt. \end{aligned}$$

Moreover, by the means value theorem, there is some $t_3 \in (t_1, t_2)$ such that

$$\int_{t_1}^{t_2} c_i(C_1(t)) C'_2(t) dt = c_i(C_1(t_3)) \int_{t_1}^{t_2} C'_2(t) dt \leq \sup_{t \in [t_1, t_2]} c_i(C_1(t)) |C_2|_{\mathcal{J},H}.$$

□

Lemma 32. *Given $M > 0$, there are constants $L', L'', L''' > 0$ such that*

$$(L' |C|_{\mathcal{J},H} - L'' |C|_{\mathcal{J},W}) \lambda^n(\rho_{\mathbf{i}}(x)) \leq |F_{\vartheta,\mathbf{i}}^n \circ C|_{\mathcal{J},H} \leq L''' \lambda^n(\rho_{\mathbf{i}}(x))$$

for any curve $C : I \rightarrow [0, 1] \times [-M, M]$, $(\vartheta, \mathbf{i}) \in [0, 1]^n \times \{0, \dots, \ell - 1\}^n$, $x \in [0, 1]$ and $n \in \mathbb{N}$.

Proof. Recall that the length of the interval J is denoted by $|J|$. By applying Lemma 31, inductively, we obtain

$$|F_{\vartheta,\mathbf{i}} \circ C|_{\mathcal{J},H} \geq \hat{c}_0 \cdots \hat{c}_{n-1} |C|_{\mathcal{J},H} - b \sum_{j=1}^n |\rho_{[\mathbf{i}]_{n-j}}(I)| \prod_{k=n-j+1}^{n-1} \hat{c}_k,$$

where $\hat{c}_k := \inf_{t \in I} c_{i_k}(\rho_{[\mathbf{i}]_k} \circ C_1(t))$ and $\prod_{k=n}^{n-1} \hat{c}_k := 1$. Let $(x_k)_{k=0}^{n-1} \subset [0, 1]$ be such that

$$\hat{c}_k = \inf_{t \in I} \lambda(\rho_{[\mathbf{i}]_{k+1}} \circ C_1(t)) = \lambda(\rho_{[\mathbf{i}]_{k+1}}(x_k)).$$

Since

$$\begin{aligned} \left| \log \prod_{k=n-j+1}^{n-1} \frac{\lambda(\rho_{[\mathbf{i}]_{k+1}}(x_k))}{\lambda(\rho_{[\mathbf{i}]_{k+1}}(x))} \right| &\leq \sum_{k=n-j+1}^{n-1} \|(\log \lambda)'\|_{\infty} |\rho_{[\mathbf{i}]_{k+1}}(x_k) - \rho_{[\mathbf{i}]_{k+1}}(x)| \\ &\leq \frac{\|(\log \lambda)'\|_{\infty}}{1 - \|1/\tau'\|_{\infty}} =: \log D_0, \end{aligned}$$

we have

$$\prod_{k=n-j+1}^{n-1} \hat{c}_k = \lambda^{j-1}(\rho_{\mathbf{i}}(x)) \prod_{k=n-j+1}^{n-1} \frac{\lambda(\rho_{[\mathbf{i}]_{k+1}}(x_k))}{\lambda(\rho_{[\mathbf{i}]_{k+1}}(x))} \in [D_0^{-1}, D_0] \cdot \lambda^{j-1}(\rho_{\mathbf{i}}(x)) \quad (14)$$

for all $x \in [0, 1]$ and $j = 2, \dots, n$. On the other hand, observe that $|\rho_{[i]_{n-j}}(I)| = \rho'_{[i]_{n-j}}(u_{n-j}) |I|$ for some $u_{n-j} \in [0, 1]$. In view of a similar estimate to the above, there is a constant $D_1 > 0$ such that

$$|\rho_{[i]_{n-j}}(I)| = (1/\tau')^{n-j} (\rho_{[i]_{n-j}}(x)) |I| \prod_{k=0}^{n-j-1} \frac{(1/\tau') (\rho_{[i]_k}(u_k))}{(1/\tau') (\rho_{[i]_k}(x))} \in [D_1^{-1}, D_1] \cdot (1/\tau')^{n-j} (\rho_{[i]_{n-j}}(x)) |I|$$

for any $x \in [0, 1]$ and $j = 1, \dots, n$, where we need to use the ε -Hölder semi-norm $|\cdot|_\varepsilon$ for a proper $\varepsilon > 0$ in order to obtain the constant $\log D_1 := |\log(1/\tau')|_\varepsilon / (1 - \|1/\tau'\|_\infty)^\varepsilon$ if the derivative of $\log(1/\tau')$ does not exist. Now, the lower bound follows as

$$\begin{aligned} |F_{\vartheta, i} \circ C|_{\mathcal{J}, H} &\geq \lambda^{n-1}(\rho_i(x)) |C|_{\mathcal{J}, H} - b D_0 D_1 |I| \sum_{j=1}^n (1/\tau')^{n-j} (\rho_{[i]_{n-j}}(x)) \lambda^{j-1}(\rho_i(x)) \\ &= \lambda^n(\rho_i(x)) \left(\frac{1}{\lambda(\rho_{i_{n-1}}(x))} |C|_{\mathcal{J}, H} - b D_0 D_1 \lambda(\rho_{[i]_{n-j}}(x)) |I| \sum_{j=0}^{n-1} (1/\tau' \lambda)^{n-j} (\rho_{[i]_{n-j}}(x)) \right) \\ &\geq \lambda^n(\rho_i(x)) \left(\inf(1/\lambda) |C|_{\mathcal{J}, H} - \frac{b D_0 D_1 \|\lambda'\|_\infty}{1 - \|1/\tau' \lambda\|_\infty} |C|_{\mathcal{J}, W} \right). \end{aligned}$$

For the upper bound, we use the following analogous relation to (14) that

$$\prod_{k=n-j+1}^{n-1} \bar{c}_k \in [D_0^{-1}, D_0] \cdot \lambda^{j-1}(\rho_i(x)),$$

for all $x \in [0, 1]$ and $j = 2, \dots, n$, where $\bar{c}_k := \sup_{t \in I} c_{i_k}(\rho_{[i]_k} \circ C_1(t))$. Similarly to the above calculations, we can derive

$$\begin{aligned} |F_{\vartheta, i} \circ C|_{\mathcal{J}, H} &\leq \lambda^{n-1}(\rho_i(x)) |C|_{\mathcal{J}, H} + b D_0 D_1 |I| \sum_{j=1}^n (1/\tau')^{n-j} (\rho_{[i]_{n-j}}(x)) \lambda^{j-1}(\rho_i(x)) \\ &\leq \lambda^n(\rho_i(x)) \left(\|1/\lambda\|_\infty + \frac{b D_0 D_1 \|\lambda'\|_\infty}{1 - \|1/\tau' \lambda\|_\infty} \right). \end{aligned}$$

□

Lemma 33. Assume that W_ϑ is non-degenerate for almost all ϑ . We define

$$a(\vartheta) := -\log \sup_{u, v \in \mathcal{J}} |W_\vartheta(u) - W_\vartheta(v)| - L \cdot |u - v|$$

for $\vartheta \in [0, 1]^{\mathbb{N}_0}$. Then, if $L > 0$ is sufficiently large, then $\lim_{n \rightarrow \infty} \frac{\log a(\sigma^n \vartheta)}{n} = 0$, almost surely.

Proof. Observe that if $\dim_H \mu_\vartheta > \dim_H \nu$, then $a(\vartheta) \in \mathbb{R}$ since W_ϑ is non-degenerate on \mathcal{J} . Thus, the function a is almost everywhere well-defined. Let $\tilde{\lambda} := \inf \lambda$ and $M := \sup_{\vartheta, u} |W_\vartheta(u)|$. Given $u, v \in [0, 1]$ and $i \in \{0, \dots, \ell - 1\}$, there is some $w \in [u, v]$ such that $|\rho_i(u) - \rho_i(v)| = \rho'_i(w) |u - v|$. By (12) we have

$$W_\vartheta(\rho_i(u)) = \lambda(\rho_i(u)) W_{\sigma\vartheta}(u) + g(\rho_i(u) + \vartheta_0).$$

Observe that

$$\begin{aligned} &|\lambda(\rho_i(u)) W_{\sigma\vartheta}(u) - \lambda(\rho_i(v)) W_{\sigma\vartheta}(v)| \\ &\geq \lambda(\rho_i(w)) |W_{\sigma\vartheta}(u) - W_{\sigma\vartheta}(v)| - M \cdot (|\lambda(\rho_i(u)) - \lambda(\rho_i(w))| + |\lambda(\rho_i(w)) - \lambda(\rho_i(v))|) \\ &\geq \lambda(\rho_i(w)) |W_{\sigma\vartheta}(u) - W_{\sigma\vartheta}(v)| - 2M \|\lambda'\|_\infty |u - v|, \end{aligned}$$

that

$$-|g(\rho_i(u) + \vartheta_0) - g(\rho_i(v) + \vartheta_0)| \geq -\|g'\|_\infty |u - v|,$$

and that

$$\lambda(\rho_i(w)) - \rho'_i(w) = \lambda(\rho_i(w)) \left(1 - \frac{1}{(\lambda\tau')(\rho_i(w))} \right) \geq \inf \lambda \left(1 - \frac{1}{\inf(\lambda\tau')} \right) =: \delta_0 > 0.$$

Thus, we have

$$\begin{aligned} & |W_{\boldsymbol{\vartheta}}(\rho_i(u)) - W_{\boldsymbol{\vartheta}}(\rho_i(v))| - L |\rho_i(u) - \rho_i(v)| \\ \geq & \lambda(\rho_i(w)) (|W_{\sigma\boldsymbol{\vartheta}}(u) - W_{\sigma\boldsymbol{\vartheta}}(v)| - L |u - v|) \\ & + (L \lambda(\rho_i(w)) - L \rho'_i(w) - 2M \|\lambda'\|_{\infty} - \|g'\|_{\infty}) |u - v| \\ \geq & \tilde{\lambda} (|W_{\sigma\boldsymbol{\vartheta}}(u) - W_{\sigma\boldsymbol{\vartheta}}(v)| - L |u - v|) + (L \delta_0 - 2M \|\lambda'\|_{\infty} - \|g'\|_{\infty}) |u - v| \\ \geq & \tilde{\lambda} (|W_{\sigma\boldsymbol{\vartheta}}(u) - W_{\sigma\boldsymbol{\vartheta}}(v)| - L |u - v|), \end{aligned}$$

if we choose $L \geq (2M \|\lambda'\|_{\infty} + \|g'\|_{\infty}) \delta_0^{-1}$. Since

$$\tilde{\lambda} e^{-a(\sigma\boldsymbol{\vartheta})} \leq \inf_{i \in \{0, \dots, \ell-1\}} \sup_{u, v \in I_i} |W_{\boldsymbol{\vartheta}}(u) - W_{\boldsymbol{\vartheta}}(v)| - L |u - v| \leq e^{-a(\boldsymbol{\vartheta})},$$

we have $a(\sigma\boldsymbol{\vartheta}) - a(\boldsymbol{\vartheta}) \geq \log \tilde{\lambda}$, from which follows $\int a(\sigma\boldsymbol{\vartheta}) - a(\boldsymbol{\vartheta}) dm^{\mathbb{N}_0}(\boldsymbol{\vartheta}) = 0$ in view of [Kel96, Lemma 2]. Thus, we have by Birkhoff's ergodic theorem that

$$\lim_{n \rightarrow \infty} \frac{a(\sigma^n \boldsymbol{\vartheta})}{n} = \frac{a(\boldsymbol{\vartheta})}{n} + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n a(\sigma^k \boldsymbol{\vartheta}) - a(\sigma^{k-1} \boldsymbol{\vartheta}) = 0$$

for almost all $\boldsymbol{\vartheta}$. □

Proof of Lemma 29. Let $L', L'', L''' > 0$ be the constant from Lemma 32 for $M := \sup_{x, \boldsymbol{\vartheta}} |W_{\boldsymbol{\vartheta}}^{\text{ext}}(x)|$, where $W_{\boldsymbol{\vartheta}}^{\text{ext}}$ is the extension of Lemma 1.

For the lower bound, let $x \in \mathcal{J}$, $n \in \mathbb{N}$ and $\boldsymbol{\vartheta} \in [0, 1]^{\mathbb{N}_0}$ be arbitrary. We define the curve $C : [0, 1] \rightarrow [0, 1] \times \mathbb{R}$ by $C(t) := (t, W_{\sigma^n \boldsymbol{\vartheta}}^{\text{ext}}(t))$. By (13), we have

$$(t, W_{\boldsymbol{\vartheta}}^{\text{ext}}(t)) = F_{[\boldsymbol{\vartheta}]_n, [x]_n}(C(\tau^n t)) \quad (15)$$

for all $t \in \mathcal{J} \cap I_n(x)$. Thus, by Lemma 32 we have

$$\begin{aligned} \sup_{v \in \mathcal{J} \cap I_n(x)} |W_{\boldsymbol{\vartheta}}(x) - W_{\boldsymbol{\vartheta}}(v)| & \leq \sup_{u, v \in \mathcal{J} \cap I_n(x)} |W_{\boldsymbol{\vartheta}}(u) - W_{\boldsymbol{\vartheta}}(v)| \\ & = |F_{[\boldsymbol{\vartheta}]_n, [x]_n} \circ C|_{\mathcal{J}, H} \leq L''' \lambda^n(\rho_{[x]_n}(\tau^n x)) = L''' \lambda^n(x). \end{aligned}$$

Therefore, the first claim is satisfied with $\overline{C} := L'''$.

In the following, we only prove the last claim since the second one follows similarly since a degenerate $W_{\boldsymbol{\vartheta}}$ is clearly non-degenerate. Suppose that $W_{\boldsymbol{\vartheta}}$ is non-degenerate for almost all $\boldsymbol{\vartheta}$. Furthermore, let $a : [0, 1]^{\mathbb{N}_0} \rightarrow \mathbb{R} \cup \{+\infty\}$ be the function of Lemma 33 with respect to a sufficiently large constant $L > L''/L'$. To end the proof, we verify that the function $c(\boldsymbol{\vartheta}) := (L'/2) e^{-a(\boldsymbol{\vartheta})}$ satisfies the assertion for the upper bound. As it is evident by definition that $\lim_{n \rightarrow \infty} \frac{\log c(\sigma^n \boldsymbol{\vartheta})}{n} = 0$ a.s., we only need to check the upper bound inequality. Let $x \in \mathcal{J}$ and $n \in \mathbb{N}$. Further, let C be the curve as above. By the continuity of $W_{\sigma^n \boldsymbol{\vartheta}}$, there are $t_1, t_2 \in \mathcal{J}$ such that

$$e^{-a(\sigma^n \boldsymbol{\vartheta})} = |W_{\sigma^n \boldsymbol{\vartheta}}(t_1) - W_{\sigma^n \boldsymbol{\vartheta}}(t_2)| - L \cdot |t_1 - t_2|.$$

Let \hat{C} be the restriction of C on the interval $[t_1, t_2]$, which forms again a curve. By Lemma 32

and (15), we have

$$\begin{aligned}
c(\sigma^n \boldsymbol{\vartheta}) \lambda^n(x) &= (L'/2) (|W_{\sigma^n \boldsymbol{\vartheta}}(t_1) - W_{\sigma^n \boldsymbol{\vartheta}}(t_2)| - L \cdot |t_1 - t_2|) \lambda^n(x) \\
&\leq 2^{-1} (L' |W_{\sigma^n \boldsymbol{\vartheta}}(t_1) - W_{\sigma^n \boldsymbol{\vartheta}}(t_2)| - L'' \cdot |t_1 - t_2|) \lambda^n(x) \\
&\leq 2^{-1} \left(L' |\hat{C}|_{\mathcal{J}, H} - L'' |\hat{C}|_{\mathcal{J}, W} \right) \lambda^n(\rho_{[x]_n}(\tau^n x)) \\
&\leq 2^{-1} \left| F_{[\boldsymbol{\vartheta}]_n, [x]_n}^n \circ \hat{C} \right|_{\mathcal{J}, H} \\
&\leq 2^{-1} \left| F_{[\boldsymbol{\vartheta}]_n, [x]_n}^n \circ C \right|_{\mathcal{J}, H} \\
&= 2^{-1} \sup_{u, v \in \mathcal{J} \cap I_n(x)} |W_{\boldsymbol{\vartheta}}(u) - W_{\boldsymbol{\vartheta}}(v)| \leq \sup_{v \in \mathcal{J} \cap I_n(x)} |W_{\boldsymbol{\vartheta}}(x) - W_{\boldsymbol{\vartheta}}(v)|.
\end{aligned}$$

□

6.3 Proofs of Theorems 2 and 3

Theorem 2 follows from Lemmas 26, 27 and 28, while Theorem 3 follows from Lemmas 26, 27 and Corollary 30.

7 Hölder spectra

Let us first consider the Housdorff spectrum of the (sub-, sup-) level sets:

$$\begin{aligned}
S_\alpha &:= \left\{ x \in [0, 1] : \liminf_{n \rightarrow \infty} \frac{-\log \lambda^n(x)}{\log |\tau'|^n(x)} = \alpha \right\}, \\
S_\alpha^{\leq} &:= \left\{ x \in [0, 1] : \liminf_{n \rightarrow \infty} \frac{-\log \lambda^n(x)}{\log |\tau'|^n(x)} \leq \alpha \right\} \quad \text{and} \quad S_\alpha^{\geq} := \left\{ x \in [0, 1] : \liminf_{n \rightarrow \infty} \frac{-\log \lambda^n(x)}{\log |\tau'|^n(x)} \geq \alpha \right\}.
\end{aligned}$$

Recall the following well-known fact, for detail consult [Pes97].

Lemma 34. *If $\mathbf{A} \neq \emptyset$, we have*

$$\mathcal{D}(\alpha) = \dim_H S_\alpha = \begin{cases} \dim_H S_\alpha^{\leq} & \text{if } \alpha \in (\alpha_{\min}, \alpha_c] \\ \dim_H S_\alpha^{\geq} & \text{if } \alpha \in [\alpha_c, \alpha_{\max}) \end{cases}$$

for $\alpha \in \mathbf{A}$. Furthermore, there exists a Gibbs measure $\nu_\alpha \in \mathcal{P}_\tau([0, 1])$ which is exact dimensional such that $\nu_\alpha(S_\alpha) = 1$ and

$$\dim_H \nu_\alpha = \dim_H S_\alpha.$$

Our results are based on the above basic fact. Indeed, in the previous sections we already showed the following. Recall that $\mathcal{N} := \bigcup_{n \in \mathbb{N}} \tau^{-n} \{0, 1\}$.

Lemma 35. *If W is non-degenerate, we have*

$$\text{hol}_W(x) = \liminf_{n \rightarrow \infty} \frac{-\log \lambda^n(x)}{\log |I_n(x)|}$$

for all $x \in \mathcal{J} \setminus \mathcal{N}$.

In addition, if $W_{\boldsymbol{\vartheta}}$ is non-degenerate for almost all $\boldsymbol{\vartheta}$, then $\text{hol}_{W_{\boldsymbol{\vartheta}}}$ satisfies the above formula for almost all $\boldsymbol{\vartheta}$.

Proof. These follow from Lemmas 17 and 29.

□

7.1 Proof of Theorem 4

It is immediate to determine the Hausdorff spectrum $\alpha \mapsto \dim_H E_\alpha$.

Proof of Theorems 4. By Lemma 35 we have $E_\alpha \triangle S_\alpha \subseteq \mathcal{N}$, and thus $\dim_H E_\alpha = \dim_H S_\alpha$ for all $\alpha \in \mathbb{R}$. In case $\mathbf{A} \neq \emptyset$, the assertion of the theorem follows from Lemma 34. In case $\mathbf{A} = \emptyset$, it follows from the same lemma that $\text{hol}_W(x) = \alpha_c$ for all $x \in \mathcal{J} \setminus \mathcal{N}$. Thus, it only remains to show $\text{hol}_W(x) = 0$ for $x \in \mathcal{J} \cap \mathcal{N}$, unless this set is empty. Recall that the case $\mathbf{A} = \emptyset$ is exactly the case when $\alpha_c \log |\tau'| + \log \lambda$ is cohomologous to 0, i.e. it is equal to $\phi \circ \tau - \phi$ for some bounded function $\phi : \mathcal{J} \rightarrow \mathbb{R}$. In particular, in view of Lemma 12, there is a constant $C_1 > 0$ such that $C_1^{-1} \leq |I_n(u)|^{-\alpha_c} \lambda^n(x) \leq C_1$ for all $u \in \mathcal{J}$ and $n \in \mathbb{N}$. Now, let $x \in \mathcal{J} \cap \mathcal{N}$ be fixed. If $x \in \{0, 1\}$, the claim follows immediately from Lemmas 16 and 29 as $B_{|I_n(0)|}(0) = \overline{I_n(0)}$. Thus we assume that $x \notin \{0, 1\}$. For sufficiently large $n \in \mathbb{N}$, we have $\tau^n x \in \partial I_n(x)$, and thus there are some $m_n \in \mathbb{N}$ and $x_n \in \mathcal{J}$ such that

$$x \in \overline{I_{m_n}(x_n)}, \quad I_n(x) \neq I_{m_n}(x_n) \quad \text{and} \quad |I_n(x)| \leq |I_{m_n}(x_n)| \leq \delta_0^{-1} |I_n(x)|,$$

where $\delta_0 > 0$ is the constant from Lemma 12. Thus we have

$$I_n(x) \subseteq B_{|I_n(x)|}(x) \subseteq I_n(x) \cup I_{m_n}(x_n).$$

By Lemma 29, there is a constant $C_2 > 0$ such that

$$\sup_{u \in B_{|I_n(x)|}(x)} |W(x) - W(u)| \in [C_2^{-1}, C_2] \cdot |I_n|^{\alpha_c}$$

for sufficiently large $n \in \mathbb{N}$. Now, the claim follows from Lemma 16. □

7.2 Proof of Theorem 5

In the following, we assume that $\boldsymbol{\vartheta}$ is i.i.i. to the uniform distribution on $[0, 1]$.

Lemma 36. *Suppose $\mathbf{A} \neq \emptyset$. Moreover, suppose that for each $\alpha \in \mathbf{A}$, it holds almost surely that*

$$E_{\boldsymbol{\vartheta}, \alpha} \triangle S_\alpha$$

is a countable set for any $\alpha \in \mathbf{A}$. Then, we have almost surely that

$$\min \left\{ \mathcal{D}(\alpha) + 1 - \alpha, \frac{\mathcal{D}(\alpha)}{\alpha} \right\} \leq \dim_H \tilde{E}_{\boldsymbol{\vartheta}, \alpha}$$

for all $\alpha \in \mathbf{A}$.

Proof. Let $\alpha \in \mathbf{A}$ and let $\tilde{S}_{\boldsymbol{\vartheta}, \alpha}$, $\tilde{S}_{\boldsymbol{\vartheta}, \alpha}^{\leq}$ and $\tilde{S}_{\boldsymbol{\vartheta}, \alpha}^{\geq}$ denote the lifts of $S_{\boldsymbol{\vartheta}, \alpha}$, $S_{\boldsymbol{\vartheta}, \alpha}^{\leq}$ and $S_{\boldsymbol{\vartheta}, \alpha}^{\geq}$ on the graph of $W_{\boldsymbol{\vartheta}}$, respectively. By Lemmas 34 there is a Gibbs measure ν_α such that $\nu_\alpha(\tilde{S}_\alpha) = 1$. Thus, if $\mu_{\boldsymbol{\vartheta}, \alpha}$ denotes the lift of ν_α on the graph of $W_{\boldsymbol{\vartheta}}$, we have almost surely $\mu_{\boldsymbol{\vartheta}, \alpha}(\tilde{S}_{\boldsymbol{\vartheta}, \alpha}) = 1$. As $\dim_H(\tilde{E}_{\boldsymbol{\vartheta}, \alpha} \triangle \tilde{S}_{\boldsymbol{\vartheta}, \alpha}) = 0$ by assumption, by Theorems 1 and 4 we have almost surely

$$\begin{aligned} \dim_H \tilde{E}_{\boldsymbol{\vartheta}, \alpha} &\geq \dim \mu_{\boldsymbol{\vartheta}, \alpha} = \min \left\{ \dim_H \nu_\alpha + 1 + \frac{\int \log \lambda d\nu_\alpha}{\int \log |\tau'| d\nu_\alpha}, \frac{h_\tau(\nu_\alpha)}{\int \log |\tau'| d\nu_\alpha} \right\} \\ &= \min \left\{ \mathcal{D}(\alpha) + 1 - \alpha, \frac{\mathcal{D}(\alpha)}{\alpha} \right\}. \end{aligned}$$

Thus, there is a set $Z \subseteq [0, 1]^{\mathbb{N}_0}$ of full probability such that

$$\dim_H \tilde{E}_{\boldsymbol{\vartheta}, \alpha} \geq \min \left\{ \mathcal{D}(\alpha) + 1 - \alpha, \frac{\mathcal{D}(\alpha)}{\alpha} \right\}$$

for all $\alpha \in A \cap \mathbb{Q}$ and $\vartheta \in Z$. Now, let $\alpha \in (\alpha_{\min}, \alpha_c]$. For any sequence $(\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{Q}$ such that $\alpha_n \nearrow \alpha$, we have by Lemma 34

$$\dim_H \tilde{E}_{\vartheta, \alpha} = \dim_H \tilde{S}_{\vartheta, \alpha} = \dim_H \tilde{S}_{\vartheta, \alpha}^{\leq} \geq \dim_H S_{\alpha_n}^{\leq} = \tilde{D}(\alpha_n) \xrightarrow{n \rightarrow \infty} \tilde{D}(\alpha)$$

for all $\vartheta \in Z$. Thus, $\dim_H \tilde{E}_{\vartheta, \alpha} \geq \tilde{D}(\alpha)$ for all $\alpha \in (\alpha_{\min}, \alpha_c]$ and $\vartheta \in Z$. Similarly, the same result can be also verified for $\alpha \in [\alpha_c, \alpha_{\max})$ by considering sequences $\alpha_n \searrow \alpha$. Thus, the proof is finished. \square

Proof of Theorem 5. The first claim for the case $A = \emptyset$ is clear in view of Theorem 4. The second claim for the case $A \neq \emptyset$ follows from Lemmas 9 and 36. \square

References

- [Bar08] Luis Barreira. *Dimension and Recurrence in Hyperbolic Dynamics*. Springer, 2008.
- [Bed89a] Tim Bedford. The box dimension of self-affine graphs and repellers. *Nonlinearity*, 2:53–71, 1989.
- [Bed89b] Tim Bedford. Hölder exponents and box dimension for self-affine fractal functions. *Constructive Approximation*, 5(1):33–48, 1989.
- [Bá15] Balázs Bárány. On the Ledrappier-Young formula for self-affine measures. *Mathematical Proceedings of the Cambridge Philosophical Society*, 2015.
- [Fal05] Kenneth Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, Ltd, 2 edition, 2005.
- [Hun98] Brian R. Hunt. The Hausdorff dimension of graphs of Weierstrass functions. *Proceedings of the American mathematical society*, 126:791–800, 1998.
- [Jin11] Xiong Jin. The graph and range singularity spectra of b-adic independent cascade functions. *Advances in Mathematics*, 226(6):4987 – 5017, 2011.
- [JS15] Johannes Jaerisch and Hiroki Sumi. Multifractal formalism for expanding rational semi-groups and random complex dynamical systems. *Nonlinearity*, 28(8):2913, 2015.
- [Kel96] Gerhard Keller. A note on strange nonchaotic attractors. *Fund. Math*, 151:139–148, 1996.
- [Kel98] Gerhard Keller. *Equilibrium States in Ergodic Theory*. London Mathematical Society Student Texts. Cambridge University Press, 1998.
- [Kel14] Gerhard Keller. An elementary proof for the dimension of the graph of the classical Weierstrass function. *preprint*, 2014.
- [MW12] A Moss and C P Walkden. The Hausdorff dimension of some random invariant graphs. *Nonlinearity*, 25:743–760, 2012.
- [Pes97] Yakov B. Pesin. *Dimension Theory in Dynamical Systems*. University of Chicago Press, 1997.
- [She15] Weixiao Shen. Hausdorff dimension of the graphs of the classical Weierstrass functions. *arXiv*, 2015.
- [Tod15] Dmitry Todorov. Hölder properties of Weierstrass-like solutions of θ -twisted cohomological equations. *arxiv*, 2015.